

# CLASSIFICATION OF FOUR DIMENSIONAL REAL LIE BIALGEBRAS OF SYMPLECTIC TYPE AND THEIR POISSON-LIE GROUPS

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## Abstract

In this paper we classify all four dimensional real Lie bialgebras of symplectic type. The classical r-matrices for these Lie bialgebras and Poisson structures on all of the related four dimensional Poisson-Lie groups are also obtained. Some new integrable models for which the Poisson-Lie group plays the role as a phase space and its dual Lie group plays the role of a symmetry group of the system, are obtained.

**keywords:** Lie bialgebra, Poisson-Lie group, Classical r-matrix, Integrable systems.

## 1 Introduction

The theory of classical integrable systems is formally related to the geometry and representation theory of Poisson-Lie groups, their Lie bialgebras [1], [2] and the corresponding classical r-matrices [3] (for a review see [4]). There is a detailed classification of r-matrices only for the complex semi-simple Lie algebras [5]. Recently, non-semisimple Lie algebras has shown to play important role in physical problems. Of course there are attempts for the classification of low dimensional non-semisimple Lie bialgebras [[6] - [11]] and Lie superbialgebras [12]. In this paper we will try to classify four dimensional real Lie bialgebras of symplectic type such that on the Lie algebras  $\mathfrak{g}$  and their duals  $\tilde{\mathfrak{g}}$  we have symplectic structures. We will also obtain the classical r-matrices and Poisson structures on the corresponding Poisson-Lie groups. The reason for obtaining this classification is that we are interested in constructing physical models in which the Lie group  $G$  (of the Lie algebra  $\mathfrak{g}$ ) plays the role of phase space and the dual Lie group  $\tilde{G}$  (of the dual Lie algebra  $\tilde{\mathfrak{g}}$ ) plays the role of symmetry group of the systems (or vice versa). The outline of the paper is as follows. In section two, we briefly review the definitions and notations. In section three, after giving the list of four dimensional real Lie algebra of symplectic type [13], [14] based in to [15] (classification of real four dimensional Lie algebras); we classify four dimensional real Lie bialgebras of symplectic type according to the method given in [12]. In section four, we determine the coboundary Lie bialgebras from the list obtained in section three. Then, in section five, using Sklyanin bracket, we calculate the Poisson structures on the Poisson-Lie groups. Also for the noncoboundary Lie bialgebras using adjoint matrices we perform the same calculations. The complete lists of compatible symplectic structures on real four dimensional Poisson-Lie groups are given in tables 8 and 9. Finally in section six, two integrable systems as examples of physical applications are obtained. For one of these examples, the Lie group  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$  plays the role of phase space and the dual Lie group  $\mathbf{A}_{4,9}^1$  plays the role of symmetry group of the systems and in the second example these roles are exchanged. Some conclusion remarks are given in section seven. Other computational results are given in appendices A-D.

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## 2 Definitions and notations

In this section, we recall some basic definitions and propositions of Lie bialgebras [1],[2] (see for a review [4] ). Let  $\mathfrak{g}$  be a finite dimensional Lie algebra and  $\mathfrak{g}^*$  be its dual space with respect to a non-degenerate canonical pairing  $(\cdot, \cdot)$ .

**Definition:** A *Lie bialgebra* is a Lie algebra  $\mathfrak{g}$  with a skew-symmetric linear map  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  such that:  
a)  $\delta$  is a one-cocycle, i.e.:

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \quad \forall X, Y \in \mathfrak{g}, \quad (1)$$

where 1 is the identity map on  $\mathfrak{g}$ .

b)  $\delta^t : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$  is a Lie bracket on  $\mathfrak{g}^*$ :

$$(\xi \otimes \eta, \delta(X)) = (\delta^t(\xi \otimes \eta), X) = ([\xi, \eta]_{\mathfrak{g}^*}, X) \quad \forall X \in \mathfrak{g}; \quad \xi, \eta \in \mathfrak{g}^*. \quad (2)$$

The Lie bialgebra defined in this way will be denoted by  $(\mathfrak{g}, \mathfrak{g}^*)$  or  $(\mathfrak{g}, \delta)$ .

**Proposition:**[11] If there exists an automorphism  $A$  of  $\mathfrak{g}$ , such that

$$\delta' = (A \otimes A) \circ \delta \circ A^{-1}, \quad (3)$$

then the one-cocycles  $\delta$  and  $\delta'$  of the Lie algebra  $\mathfrak{g}$  are equivalent. In this case the two Lie bialgebras  $(\mathfrak{g}, \delta)$  and  $(\mathfrak{g}, \delta')$  are equivalent.

**Definition:** Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$  is called a *coboundary* Lie bialgebra if there exists an element  $r \in \mathfrak{g} \otimes \mathfrak{g}$  such that:

$$\delta(X) = [1 \otimes X + X \otimes 1, r] \quad \forall X \in \mathfrak{g}. \quad (4)$$

**Definition:** Coboundary Lie bialgebras can be one of the two following different types:

a) If  $r$  is a skew-symmetric solution of the classical Yang-Baxter equation (CYBE):

$$[[r, r]] = 0, \quad (5)$$

then the coboundary Lie bialgebra is said to be triangular; where the Schouten bracket is defined by:

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \quad (6)$$

such that for  $r = r^{ij} X_i \otimes X_j$ , we have  $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$ ,  $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$  and  $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$ , in which  $\{X_i\}$  is the basis of the Lie algebra  $\mathfrak{g}$ .

b) If  $r$  is a solution of CYBE, such that  $r_{12} + r_{21}$  is a  $\mathfrak{g}$  invariant element of  $\mathfrak{g} \otimes \mathfrak{g}$ ; then, the coboundary Lie bialgebra is said to be quasi-triangular. Sometimes, the condition b) can be replaced with the following one [2], [16]:

b') If  $r$  is a skew-symmetric solution of the modified CYBE:

$$[[r, r]] = \omega \quad \omega \in \wedge^3 \mathfrak{g}, \quad (7)$$

then coboundary Lie bialgebra is said to be quasi-triangular. We note that if  $\mathfrak{g}$  is a Lie bialgebra then  $\mathfrak{g}^*$  is also a Lie bialgebra [16], while, this is not always true for the coboundary property.

**Definition:** *Manin triple* is a triple of Lie algebras  $(\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$  with a non-degenerate ad-invariant symmetric bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D}$ , so that

1.  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are Lie subalgebras of  $\mathcal{D}$ ,
2.  $\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$  as a vector space,
3.  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  are isotropic with respect to  $\langle \cdot, \cdot \rangle$ , i.e.,

$$\langle X_i, X_j \rangle = \langle \tilde{X}^i, \tilde{X}^j \rangle = 0, \quad \langle X_i, \tilde{X}^j \rangle = \delta^j_i. \quad (8)$$

There is a one-to-one correspondence between Manin triple  $(\mathcal{D}, \mathfrak{g}, \tilde{\mathfrak{g}})$  with  $\tilde{\mathfrak{g}} = \mathfrak{g}^*$  and Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$  [16]. If we choose the structure constants of Lie algebras  $\mathfrak{g}$  and  $\tilde{\mathfrak{g}}$  as follows:

$$[X_i, X_j] = f_{ij}^k X_k, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}_k \tilde{X}^k, \quad (9)$$

with them satisfying the following Jacobi identities;

$$f_{ij}{}^k f_{km}{}^n + f_{ik}{}^n f_{mj}{}^k + f_{jk}{}^n f_{im}{}^k = 0, \quad (10)$$

$$\tilde{f}^{ij}{}_k \tilde{f}^{km}{}_n + \tilde{f}^{im}{}_k \tilde{f}^{jk}{}_n + \tilde{f}^{jm}{}_k \tilde{f}^{ki}{}_n = 0, \quad (11)$$

then, ad-invariance of the bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$  implies that [16]-[4]

$$[X_i, \tilde{X}^j] = \tilde{f}^{jk}{}_i X_k + f_{ki}{}^j \tilde{X}^k, \quad (12)$$

where, using (8), (9) and (2) we obtain

$$\delta(X_i) = \tilde{f}^{jk}{}_i X_j \otimes X_k. \quad (13)$$

By applying the above relation in (1), one can obtain the following mixed Jacobi relation:

$$f_{kl}{}^m \tilde{f}^{ij}{}_m = f_{mk}{}^i \tilde{f}^{jm}{}_l - f_{ml}{}^i \tilde{f}^{jm}{}_k - f_{mk}{}^j \tilde{f}^{im}{}_l + f_{ml}{}^j \tilde{f}^{im}{}_k. \quad (14)$$

This relation can also be obtained from Jacobi identity of  $\mathcal{D}$ .

### 3 Classification of four dimensional real Lie bialgebras of symplectic type.

In this section, we classify four dimensional real Lie bialgebras of symplectic type, such that we have symplectic structures on the Lie algebras  $\mathfrak{g}$  and their duals  $\tilde{\mathfrak{g}}$ . The four dimensional real Lie algebras of symplectic type have been classified in [13]. Here, we will use the same method which previously have been considered in the classification of Lie superbialgebras in [12]. Let us first have a short review about symplectic structures on a Lie algebras and four dimensional real Lie algebras of symplectic type and then have a short review about classification method of low dimensional real Lie bialgebras.

#### 3.1 Four dimensional real Lie algebras of symplectic type

A symplectic structure  $\omega$  on a 2n-dimensional Lie algebra  $\mathfrak{g}$  is defined as a two form such that <sup>1</sup>

1)  $\omega$  is closed, i.e.,  $d\omega = 0$ ;

2)  $\omega$  has maximal rank, that is,  $\omega^n$  is a volume form on the corresponding Lie group. The list of four dimensional real Lie algebras with symplectic structure is given in [13] (see also [14]); and we brought it in table 1 for self containing of the paper <sup>2</sup>

**Table 1:** Four dimensional real Lie algebras of symplectic type.

$\mathfrak{g}$	Non-zero commutation relations	$\mathfrak{g}$	Non-zero commutation relations
$A_{4,1}$	$[X_2, X_4] = X_1, [X_3, X_4] = X_2$	$A_{4,2}^{-1}$	$[X_1, X_4] = -X_1, [X_2, X_4] = X_2, [X_3, X_4] = X_2 + X_3$
$A_{4,3}$	$[X_1, X_4] = X_1, [X_3, X_4] = X_2$	$A_{4,5}^{-1,-1}$	$[X_1, X_4] = X_1, [X_2, X_4] = -X_2, [X_3, X_4] = -X_3$
$A_{4,5}^{-1,b}$	$[X_1, X_4] = X_1, [X_2, X_4] = -X_2, [X_3, X_4] = b X_3$	$A_{4,5}^{a,-1}$	$[X_1, X_4] = X_1, [X_2, X_4] = a X_2, [X_3, X_4] = -X_3$
$A_{4,5}^{a,-a}$	$[X_1, X_4] = X_1, [X_2, X_4] = a X_2, [X_3, X_4] = -a X_3$	$A_{4,6}^{a,0}$	$[X_1, X_4] = a X_1, [X_2, X_4] = -X_3, [X_3, X_4] = X_2$
$A_{4,7}$	$[X_1, X_4] = 2X_1, [X_2, X_3] = X_1, [X_2, X_4] = X_2$ $[X_3, X_4] = X_2 + X_3$	$A_{4,9}^0$	$[X_1, X_4] = X_1, [X_2, X_3] = X_1, [X_2, X_4] = X_2$
$A_{4,9}^1$	$[X_1, X_4] = 2X_1, [X_2, X_3] = X_1, [X_2, X_4] = X_2$ $[X_3, X_4] = X_3$	$A_{4,9}^{-1/2}$	$[X_1, X_4] = 1/2 X_1, [X_2, X_3] = X_1, [X_2, X_4] = X_2$ $[X_3, X_4] = -1/2 X_3$
$A_{4,11}^b$	$[X_1, X_4] = 2a X_1, [X_2, X_3] = X_1$ $[X_2, X_4] = a X_2 - X_3, [X_3, X_4] = X_2 + a X_3$	$A_{4,9}^b$	$[X_1, X_4] = (1+b) X_1, [X_2, X_3] = X_1, [X_2, X_4] = X_2$ $[X_3, X_4] = b X_3$
$A_2 \oplus A_2$	$[X_1, X_2] = X_2, [X_3, X_4] = X_4$	$A_{4,12}$	$[X_1, X_3] = X_1, [X_1, X_4] = -X_2, [X_2, X_3] = X_2$ $[X_2, X_4] = X_1$
$VI_0 \oplus R$	$[X_1, X_3] = X_2, [X_2, X_3] = X_1$	$III \oplus R$	$[X_1, X_2] = -X_2 - X_3, [X_1, X_3] = -X_2 - X_3$
$VII_0 \oplus R$	$[X_1, X_3] = -X_2, [X_2, X_3] = X_1$	$II \oplus R$	$[X_2, X_3] = X_1$

<sup>1</sup>Here we use the cohomology of a Lie algebra such that  $d$  is a extrinsic derivative on the Lie algebra  $\mathfrak{g}$  (see for example [17]).

<sup>2</sup>Note that in table 1 we use the Patera classification [15] of four dimensional real Lie algebras.

### 3.2 Review of the classification method for low dimensional real Lie bialgebras

In this section, we review the method of obtaining and classification of low dimensional real Lie bialgebras, as applied for first time in [12] for classification of real low dimensional Lie superbialgebras. For this proposes, we use the following adjoint representation:

$$\begin{aligned} (\mathcal{X}_i)_j^k &= -f_{ij}^k, & (\mathcal{Y}^k)_{ij} &= -f_{ij}^k, \\ (\tilde{\mathcal{X}}^i)_k^j &= -\tilde{f}^{ij}_k, & (\tilde{\mathcal{Y}}_k)^{ij} &= -\tilde{f}^{ij}_k, \end{aligned} \quad (15)$$

for writing the matrix forms of equations (11) and (14) as follows, respectively,

$$(\tilde{\mathcal{X}}^i)_k^j \tilde{\mathcal{X}}^k + \tilde{\mathcal{X}}^i \tilde{\mathcal{X}}^j - \tilde{\mathcal{X}}^j \tilde{\mathcal{X}}^i = 0, \quad (16)$$

$$(\tilde{\mathcal{X}}^i)_l^j \mathcal{Y}^l = -(\tilde{\mathcal{X}}^t)^j \mathcal{Y}^i + \mathcal{Y}^j \tilde{\mathcal{X}}^i - \mathcal{Y}^i \tilde{\mathcal{X}}^j + (\tilde{\mathcal{X}}^t)^i \mathcal{Y}^j. \quad (17)$$

Having the structure constants of the Lie algebra  $\mathfrak{g}$  ( $f_{ij}^k$ ), we solve matrix equations (16) and (17) in order to obtain the structure constants of the dual Lie algebras  $\tilde{\mathfrak{g}}$  ( $\tilde{f}^{ij}_k$ ), such that  $(\mathfrak{g}, \tilde{\mathfrak{g}})$  is a Lie bialgebra. By this method we will classify four dimensional real Lie bialgebras of symplectic type. We will perform this task in the following three steps similar to [12].

**Step1:** Solving equations (16) and (17) and determining the Lie algebras  $\mathfrak{g}'$  which are isomorphic with dual solutions  $\tilde{\mathfrak{g}}$ .

With the solution of matrix equations (16) and (17) for obtaining matrices  $\tilde{\mathcal{X}}^i$ , some structure constants of  $\tilde{\mathfrak{g}}$  are obtained to be zero, some unknown and some obtained in terms of each other. In order to know whether  $\tilde{\mathfrak{g}}$  is one of the Lie algebras of table or isomorphic to them, we must use the following isomorphic relation between the obtained Lie algebras  $\tilde{\mathfrak{g}}$  and one of the Lie algebras of table 1, e.g.,  $\mathfrak{g}'$ . Applying the following transformation for a change of basis  $\tilde{\mathfrak{g}}$  as follows

$$\tilde{\mathcal{X}}'^i = C^i_j \tilde{\mathcal{X}}^j, \quad [\tilde{\mathcal{X}}'^i, \tilde{\mathcal{X}}'^j] = \tilde{f}'^{ij}_k \tilde{\mathcal{X}}'^k, \quad (18)$$

we have the following matrix equations for isomorphism:

$$C(C^i_j \tilde{\mathcal{X}}^j_{(\tilde{\mathfrak{g}})}) = \tilde{\mathcal{X}}^i_{(\mathfrak{g}')} C. \quad (19)$$

Solving (19) with the condition  $\det C \neq 0$ , we obtain some extra conditions on  $\tilde{f}^{kl}_{(\tilde{\mathfrak{g}})m}$ 's as imposed by (16) and (17).

**Step2:** Calculate the general form of the transformation matrices  $B : \mathfrak{g}' \rightarrow \mathfrak{g}'$ ; such that  $(\mathfrak{g}, \mathfrak{g}'.i)$  are Lie bialgebras.

As the second step we transform Lie bialgebra  $(\mathfrak{g}, \tilde{\mathfrak{g}})$  to Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}'.i)$  (where  $\mathfrak{g}'.i$  is isomorphic as Lie algebra  $\mathfrak{g}'$ ) with an automorphism of the Lie algebra  $\mathfrak{g}$ . As inner product (8) is invariant, we have  $A^{-t} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}'.i$

$$X'_i = A_i^k X_k, \quad \tilde{\mathcal{X}}'^j = (A^{-t})^j_l \tilde{\mathcal{X}}^l, \quad \langle X'_i, \tilde{\mathcal{X}}'^j \rangle = \delta^j_i, \quad (20)$$

where  $A^{-t}$  is the inverse transpose of same matrix  $A \in \text{Aut}(\mathfrak{g})$ . Thus, we have the following relation

$$(A^{-t})^i_k \tilde{f}^{kl}_{(\tilde{\mathfrak{g}})m} (A^{-t})^j_l = f^{ij}_{(\mathfrak{g}'.i)n} (A^{-t})^n_m. \quad (21)$$

Now, for derivation of Lie bialgebras  $(\mathfrak{g}, \mathfrak{g}'.i)$ , we must find the Lie algebras  $\mathfrak{g}'.i$  or transformations  $B : \mathfrak{g}' \rightarrow \mathfrak{g}'.i$ , such that

$$B^i_k f^{kl}_{(\mathfrak{g}')m} B^j_l = f^{ij}_{(\mathfrak{g}'.i)n} B^n_m. \quad (22)$$

For this purpose, it is enough to eliminate  $f^{ij}_{(\mathfrak{g}'.i)n}$  between (21) and (22). Then, we have the following matrix equation for  $B$ :

$$(A^{-t})^i_m \tilde{\mathcal{X}}'^t_m A^{-1} = (B^t A)^{-1} (B^i_k \mathcal{X}^t_k) B^t. \quad (23)$$

Now, by solving (23), we obtain the general form of the matrix  $B$  with the condition  $\det B \neq 0$ .

**Step 3:** Calculate and classify the non-equivalent Lie bialgebras.

Having solved (23), we obtain the general form of the matrix  $B$  so that its elements are written in terms of the elements of matrices  $A, C$  and structure constants  $\tilde{f}_{(\tilde{\mathbf{g}})}^{ij}$ . Now after substituting  $B$  in (22), we obtain the structure constants  $f_{(\mathbf{g}'.i)}^{ij}$  of the Lie algebra  $\mathbf{g}'.i$  in terms of elements of matrices  $A$  and  $C$  and structure constants  $\tilde{f}_{(\tilde{\mathbf{g}})}^{ij}$ . Then, it is checked whether it is possible to equalize the structure constants  $f_{(\mathbf{g}'.i)}^{ij}$  with each other and with  $\pm 1$  or not, such that  $\det B \neq 0, \det A \neq 0$  and  $\det C \neq 0$ . In this way, we obtain matrices  $B_1, B_2$ , etc. Note that in obtaining  $B_i$ s we impose the condition  $BB_i^{-1} \in \text{Aut}^t(\mathbf{g})$ . If this condition is not satisfied then we cannot impose it on the structure constants, because  $B$  and  $B_i$  are not equivalent (see below). Now, using isomorphism of matrices  $B_1, B_2$ , etc., we can obtain Lie bialgebras  $(\mathbf{g}, \mathbf{g}'.i), (\mathbf{g}, \mathbf{g}'.ii)$ , etc. On the other hand, there remains a question of whether these Lie bialgebras are equivalent? In order to answer this question, we use the matrix form of relation (3). Consider the two Lie bialgebras  $(\mathbf{g}, \mathbf{g}'.i), (\mathbf{g}, \mathbf{g}'.ii)$ ; then using

$$A(X_i) = A_i^j X_j, \quad (24)$$

the relation (3) will have the following matrix form:

$$A^t((A^t)^i_k \mathcal{X}_{(\mathbf{g}'.i)}^k) = \mathcal{X}_{(\mathbf{g}'.ii)}^i A^t. \quad (25)$$

On the other hand, the transformation matrix between  $\mathbf{g}'.i$  and  $\mathbf{g}'.ii$  is  $B_2 B_1^{-1}$  such that  $B_1 : \mathbf{g}' \rightarrow \mathbf{g}'.i$  and  $B_2 : \mathbf{g}' \rightarrow \mathbf{g}'.ii$ , then we have

$$(B_2 B_1^{-1})((B_2 B_1^{-1})^i_k \mathcal{X}_{(\mathbf{g}'.i)}^k) = \mathcal{X}_{(\mathbf{g}'.ii)}^i (B_2 B_1^{-1}). \quad (26)$$

A comparison of (26) with (25) reveals that if  $B_2 B_1^{-1} \in A^t$ , then, the Lie bialgebras  $(\mathbf{g}, \mathbf{g}'.i)$  and  $(\mathbf{g}, \mathbf{g}'.ii)$  are equivalent. In this way, we obtain non-equivalent class of  $B_i$ s, and we consider only one element of this class. Now, we will use this method for obtaining and classifying all four dimensional real Lie bialgebras of symplectic type.

### 3.3 Classification of four dimensional real Lie bialgebra of symplectic type

In the following we explain the above method by describing the details of the calculations for obtaining the symplectic Lie bialgebra  $(A_{4,1}, (II \oplus R).i)$ .

#### An example:

One of the solutions of Jacobi and mixed Jacobi identities (16) and (17) for the Lie algebra  $A_{4,1}$  has the following form:

$$\tilde{f}_{2}^{12} = \tilde{f}_{4}^{14} = \alpha, \quad \tilde{f}_{3}^{12} = \beta, \quad \tilde{f}_{4}^{12} = \gamma, \quad \tilde{f}_{4}^{23} = \lambda, \quad (27)$$

where  $\alpha, \beta, \gamma, \lambda$  are arbitrary constants. Now, using (19) the corresponding Lie algebra  $\tilde{\mathbf{g}}$  is isomorphic to the Lie algebra  $II \oplus R$ , with the following isomorphism matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ \frac{-c_{13}c_{22}+c_{12}c_{23}-c_{41}\gamma}{\beta} & 0 & 0 & c_{34} \\ c_{41} & 0 & 0 & -\frac{c_{34}\beta}{\gamma} \end{pmatrix}, \quad (28)$$

satisfying the conditions  $\alpha = 0$  and  $\lambda = 0$ , where,  $c_{ij} \in \mathfrak{R}$  are arbitrary elements of the isomorphism matrices. Now, by substituting the above results and using the following automorphism group element of the Lie algebra  $A_{4,1}$ , we have

$$A = \begin{pmatrix} a_{22}a_{44} & 0 & 0 & 0 \\ a_{32}a_{44} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & \frac{a_{22}}{a_{44}} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad (29)$$

where  $a_{22}a_{44} \neq 0$  and  $a_{ij}$  are arbitrary elements of the  $A$  matrices. In (23), one can obtain the following form for the matrix  $B$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \frac{-a_{43}a_{44}b_{41}\beta - a_{22}^2a_{44}^2(b_{13}b_{22} - b_{12}b_{23}) - a_{44}^2b_{41}\gamma}{a_{22}\beta} & 0 & 0 & -\frac{a_{43}a_{44}b_{44}\beta + a_{44}^2b_{44}\gamma}{a_{22}\beta} \\ b_{41} & 0 & 0 & b_{44} \end{pmatrix}, \quad (30)$$

where  $b_{ij}$  are arbitrary elements of the  $B$  matrices. Now, using (22) we have found the following commutation relations for the algebra  $\mathfrak{g}' \cdot i$

$$[\tilde{X}^1, \tilde{X}^2] = \frac{\beta}{a_{22} a_{44}^2} \tilde{X}^3 + \frac{a_{43} \beta + a_{44} \gamma}{a_{22}^2 a_{44}} \tilde{X}^4. \quad (31)$$

On the other hand, by choosing  $a_{43} = 0$ , we have

$$B_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \frac{-a_{22}^2a_{44}^2(b_{13}b_{22} - b_{12}b_{23}) - a_{44}^2b_{41}\gamma}{a_{22}\beta} & 0 & 0 & -\frac{a_{44}^2b_{44}\gamma}{a_{22}\beta} \\ b_{41} & 0 & 0 & b_{44} \end{pmatrix}, \quad (32)$$

where, since  $B_1 B_1^{-1} \in A^t$ , then,  $B_1$  is equivalent to  $B$ . Furthermore, by choosing  $\frac{\beta}{a_{22} a_{44}^2} = \frac{\gamma}{a_{22}^2} = q \neq 0$ , we have

$$B_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \frac{-(b_{13}b_{22} - b_{12}b_{23})}{q} - b_{41} & 0 & 0 & -b_{44} \\ b_{41} & 0 & 0 & b_{44} \end{pmatrix}, \quad (33)$$

where, since  $B_2 B_1^{-1} \in A^t$ , then,  $B_2$  is equivalent to  $B_1$ . Now, by choosing  $q = 1$ , we have

$$B_3 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ -(b_{13}b_{22} - b_{12}b_{23}) - b_{41} & 0 & 0 & -b_{44} \\ b_{41} & 0 & 0 & b_{44} \end{pmatrix}, \quad (34)$$

where, since  $B_3 B_2^{-1} \in A^t$ , then,  $B_3$  is equivalent to  $B_2$ . So, we obtained the Lie bialgebra  $(A_{4,1}, (II \oplus R) \cdot i)$  such that commutation relations for  $(II \oplus R) \cdot i$  are as follows:

$$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3 + \tilde{X}^4. \quad (35)$$

Similarly, we use the above method for the classification of the four dimensional real Lie bialgebras of symplectic type<sup>3</sup>. The results of such calculations are give in table 2 (Appendix A).

## 4 Four dimensional coboundary real Lie bialgebras of symplectic type

In this section, we determine number of four dimensional real Lie bialgebras of symplectic type of table 2 which are coboundary. For this purpose, we must find  $r = r^{ij} X_i \otimes X_j \in \mathfrak{g} \otimes \mathfrak{g}$  such that the cocommutator of symplectic Lie bialgebras can be written as (4). Using (4), (9) and (13), we have [11]

$$\tilde{\mathcal{Y}}_i = \mathcal{X}_i^t r + r \mathcal{X}_i. \quad (36)$$

Now using the above relations, one can find the r-matrix of the Lie bialgebras of table 2 (if there exists). One can also perform this task for the dual Lie bialgebras  $(\tilde{\mathfrak{g}}, \mathfrak{g})$  using the following equations (36)

$$\mathcal{Y}^i = (\tilde{\mathcal{X}}^i)^t \tilde{r} + \tilde{r} \tilde{\mathcal{X}}^i. \quad (37)$$

---

<sup>3</sup>Note that we consider only Lie bialgebras for which there are symplectic structures on the Lie algebra  $\mathfrak{g}$  and their duals  $\tilde{\mathfrak{g}}$ , according to table 1

The results are summarized in tables 3 and 4. Notice that, we also determine the Schouten brackets of these r-matrices and consider the classical r-matrices, satisfying the CYB equation. In table 3 of Appendix B, we have listed the triangular Lie bialgebras  $(\mathbf{g}, \tilde{\mathbf{g}})$  with triangular duals  $(\tilde{\mathbf{g}}, \mathbf{g})$ . Since such structures can be specified (up to automorphism) by pairs of r-matrices, then it is natural to call them symplectic bi-r-matrix Lie bialgebras [11]. Furthermore, we provide complete list of four dimensional triangular Lie bialgebras of symplectic type in the table 4 (Appendix B).

## 5 Calculation of Poisson and symplectic structures on four dimensional real Poisson-Lie groups

### 5.1 Calculation of Poisson and symplectic structures by Sklyanin bracket

For the Lie bialgebras one can obtain the corresponding Poisson-Lie groups by means of Sklyanin bracket provided by a given skew-symmetric r-matrix  $r = r^{ij} X_i \otimes X_j$  [16] as follows:

$$\{f_1, f_2\} = \sum_{i,j} r^{ij} ((X_i^L f_1)(X_j^L f_2) - (X_i^R f_1)(X_j^R f_2)), \quad \forall f_1, f_2 \in C^\infty(G), \quad (38)$$

where  $(X_i^L)$  and  $(X_i^R)$  are left and right invariant vector fields on the Poisson-Lie group  $G$ . For calculation of the left and right invariant vector fields on the Lie group  $G$ , it is enough to determine the left and right invariant one forms. For  $\forall g \in G$ , we have

$$dg g^{-1} = R^i X_i \quad (dg g^{-1})^i = R^i = R^i_j dx^j, \quad (39)$$

$$g^{-1} dg = L^i X_i \quad (g^{-1} dg)^i = L^i = L^i_j dx^j, \quad (40)$$

where  $x^i$  are coordinates on the Lie group. Now, from  $\delta_j^i = \langle X_j^R, R^i \rangle$  and  $\delta_j^i = \langle X_j^L, L^i \rangle$  together with  $X_j^R = X_j^{Rl} \partial_l$  and  $X_j^L = X_j^{Ll} \partial_l$ , we have

$$X_j^{Rl} = (R^{-t})_j^l, \quad X_j^{Ll} = (L^{-t})_j^l. \quad (41)$$

To calculate the above matrices, we assume the following parameterization of the four dimensional real Lie group  $G$ :

$$g = e^{x_1 X_1} e^{x_2 X_2} e^{x_3 X_3} e^{x_4 X_4}. \quad (42)$$

Then, for left and right invariant Lie algebra valued one forms, we have:

$$dg g^{-1} = dx_1 X_1 + dx_2 e^{x_1 X_1} X_2 e^{-x_1 X_1} + dx_3 e^{x_1 X_1} (e^{x_2 X_2} X_3 e^{-x_2 X_2}) e^{-x_1 X_1} \\ + dx_4 e^{x_1 X_1} e^{x_2 X_2} (e^{x_3 X_3} X_4 e^{-x_3 X_3}) e^{-x_2 X_2} e^{-x_1 X_1}, \quad (43)$$

$$g^{-1} dg = dx_1 e^{-x_4 X_4} e^{-x_3 X_3} (e^{-x_2 X_2} X_1 e^{x_2 X_2}) e^{x_3 X_3} e^{x_4 X_4} + dx_2 e^{-x_4 X_4} (e^{-x_3 X_3} X_2 e^{x_3 X_3}) e^{x_4 X_4} \\ + dx_3 e^{-x_4 X_4} X_3 e^{x_4 X_4} + dx_4 X_4, \quad (44)$$

such that, for this calculation one can use the following relation

$$(e^{-x_i X_i} X_j e^{x_i X_i}) = (e^{x_i X_i})^k_j X_k, \quad (45)$$

in which we have a summation over the index  $k$  on the right hand side. In this way, we obtained the left and right invariant vector fields as given in table 5 (Appendix C). Now, by using these results one can calculate the Poisson structures over the Lie group  $G$ . For simplicity we can rewrite the relation (38) in the following matrix form:

$$\{f_1, f_2\} = \begin{pmatrix} X_1^L f_1 & X_2^L f_1 & X_3^L f_1 & X_4^L f_1 \end{pmatrix} r \begin{pmatrix} X_1^L f_2 & X_2^L f_2 & X_3^L f_2 & X_4^L f_2 \end{pmatrix}^t \\ - \begin{pmatrix} X_1^R f_1 & X_2^R f_1 & X_3^R f_1 & X_4^R f_1 \end{pmatrix} r \begin{pmatrix} X_1^R f_2 & X_2^R f_2 & X_3^R f_2 & X_4^R f_2 \end{pmatrix}^t. \quad (46)$$

Using the tables 5 and r-matrices presented in tables 3 and 4, one can calculate Poisson brackets on the Poisson-Lie groups  $G$  and  $\tilde{G}$ . The results are listed in table 6 (Appendix D).

## 5.2 Calculation of Poisson and symplectic structures on four dimensional real Poisson-Lie group by Manin triple and map $\pi(g)$ between the Lie subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ .

Some of the Lie bialgebras in table 2 do not have r-matrix, such that, for obtaining the corresponding Poisson-Lie group one can use the following relation:

$$g^{-1}X_i g = a(g)_i^j X_j, \quad g^{-1}\tilde{X}^i g = b(g)^{ij} X_j + d(g)^i_j \tilde{X}^j, \quad (47)$$

$$\pi(g) = b(g)a^{-1}(g), \quad (48)$$

where  $\pi(g)$  is the algebraic Poisson structure, such that, the Poisson structure on the Lie group  $G$  can be obtained as follows:

$$P_G(g) = (-b(g)a(g)^{-1})^{ij} X_i^R \wedge X_j^R. \quad (49)$$

Therefore, using (41), we have the following relation for the Poisson structures

$$P^{kl} = (-b(g)a(g)^{-1})^{ij} X_i^R{}^k X_j^R{}^l. \quad (50)$$

Now, using the right invariant vector fields of table 5, and calculating the adjoint matrices  $a(g)$  and  $b(g)$ , one can obtain the Poisson brackets on the non-coboundary Lie bialgebras <sup>4</sup>. The results are given in table 7 (Appendix D).

## 5.3 Compatible symplectic structure on four dimensional real Poisson-Lie groups

Among Poisson structures given in tables 6 and 7, one can consider those with invertible matrices  $P^{ij}$ . In this way, one arrives at the compatible symplectic structure on the four dimensional real Poisson-Lie groups. Such results are given in tables 8 and 9.

**Table 8:** Related Lie bialgebras with compatible bi-symplectic<sup>5</sup> structure on four dimensional real Poisson-Lie groups

$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$
$(A_{4,7}, A_{4,7}.i)$	$(A_{4,7}, A_{4,9}^{\frac{-1}{2}}.i)$	$(A_{4,9}^0, A_{4,9}^0.iv)$	$(A_{4,9}^0, A_{4,9}^0.v)$	$(A_{4,9}^{\frac{-1}{2}}, A_{4,9}^{\frac{-1}{2}}.ii)$
$(A_{4,9}^{\frac{-1}{2}}, A_{4,9}^{\frac{-1}{2}}.iii)$	$(A_{4,9}^{\frac{-1}{2}}, A_{4,9}^{\frac{-1}{2}}.iv)$	$(A_{4,9}^1, A_{4,9}^1.i)$	$(A_{4,9}^b, A_{4,9}^b.i)$	$(A_{4,9}^{\frac{-1}{2}}, A_{4,9}^1.ii)$
$(A_{4,11}^b, A_{4,11}^b.i)$	$(A_{4,12}, A_{4,12}.i)$	$(A_{4,12}, A_{4,12}.ii)$	$(A_{4,12}, A_{4,12}.iii)$	$(A_2 \oplus A_2, (A_2 \oplus A_2).iv)$
$(A_2 \oplus A_2, (A_2 \oplus A_2).v)$	$(A_2 \oplus A_2, (A_2 \oplus A_2).vi)$	$(A_2 \oplus A_2, (A_2 \oplus A_2).vii)$		

**Table 9:** Related Lie bialgebras with compatible symplectic structure on four dimensional real Poisson-Lie groups

$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$	$(\mathfrak{g}, \tilde{\mathfrak{g}})$
$(A_{4,1}, A_{4,9}^0.i)$	$(A_{4,1}, A_{4,9}^0.ii)$	$(A_{4,1}, A_{4,9}^0.iii)$	$(A_{4,3}, (A_2 \oplus A_2).i)$
$(A_{4,3}, (A_2 \oplus A_2).iii)$	$((II \oplus R).iv, A_{4,7})$	$((II \oplus R).ii, A_{4,7})$	$((II \oplus R).x, A_{4,9}^0)$
$((II \oplus R).iv, A_{4,9}^0)$	$((II \oplus R).ii, A_{4,9}^0)$	$((II \oplus R).viii, A_{4,9}^0)$	$((II \oplus R).xi, A_{4,9}^0)$
$((II \oplus R).ii, A_{4,9}^{\frac{-1}{2}})$	$((II \oplus R).ii, A_{4,9}^1)$	$((II \oplus R).iv, A_{4,9}^b)$	$((II \oplus R).ii, A_{4,11}^b)$
$((II \oplus R).xii, A_{4,12})$	$((II \oplus R).xiii, A_2 \oplus A_2)$	$((III \oplus R).i, A_2 \oplus A_2)$	$((III \oplus R).iii, A_2 \oplus A_2)$
$((III \oplus R).v, A_2 \oplus A_2)$	$((VI_0 \oplus R).vi, A_2 \oplus A_2)$	$((VI_0 \oplus R).vii, A_2 \oplus A_2)$	$((VI_0 \oplus R).viii, A_2 \oplus A_2)$
$((VI_0 \oplus R).v, A_{4,9}^0)$	$((VII_0 \oplus R).ii, A_{4,9}^0)$	$((II \oplus R).iv, A_{4,9}^{\frac{-1}{2}})$	$((II \oplus R).xii, A_{4,12})$
$((III \oplus R).iv, A_2 \oplus A_2)$	$((VI_0 \oplus R).iv, A_{4,9}^0)$	$(A_{4,3}, (A_2 \oplus A_2).ii)$	

## 6 Physical application; example of integrable systems by use of symplectic Lie bialgebra

Here, we consider some integrable systems obtained by using the symplectic Lie bialgebras. In these examples, we consider the Lie group  $G$  related to the Lie bialgebra  $(\mathfrak{g}, \tilde{\mathfrak{g}})$  as a phase space and its dual Lie group  $\tilde{G}$  as a

<sup>4</sup>Note that using the relation (50) for coboundary Lie bialgebras one can obtain the same results which are obtained from Sklyanin bracket (38)

<sup>5</sup>By means of bi-symplectic structure is that there are symplectic structure on the Lie groups  $G$  and  $\tilde{G}$  related to Lie bialgebras  $(\mathfrak{g}, \tilde{\mathfrak{g}})$  and  $(\tilde{\mathfrak{g}}, \mathfrak{g})$  respectively.



symmetry group of the system. For this propose we use the formalism mentioned in [18] for calculation of two integrable systems with some symmetry groups  $\tilde{G}$ . Here the Lie groups  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$  and  $\mathbf{A}_{4,9}^{\mathbf{i},\mathbf{ii}}$  play the role of phase space and symmetry group, respectively.

**Example 1)** The Lie group  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$  as a phase space and  $\mathbf{A}_{4,9}^{\mathbf{i},\mathbf{ii}}$  as symmetry group :  
For this example, the Darboux coordinates have the following forms [18]:

$$y_1 = x_2, \quad y_2 = \frac{e^{-x_4}(x_1 + x_2x_3)}{x_2}, \quad y_3 = \frac{e^{-\frac{x_4}{2}}(-x_1 + e^{\frac{x_4}{2}}x_1 - x_2x_3)}{2x_2^2}, \quad y_4 = e^{\frac{x_4}{2}}, \quad (51)$$

such that in this Darboux coordinates the symplectic structure of  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$  (according table 6)

$$\{x_1, x_2\} = -2x_2^2, \quad \{x_1, x_3\} = -x_1 + x_2x_3, \quad \{x_1, x_4\} = 2x_2, \quad \{x_2, x_3\} = -2x_2, \quad \{x_3, x_4\} = -2 + 2e^{x_4/2}, \quad (52)$$

can be simplified as follows:

$$\{y_1, y_3\} = 1, \quad \{y_2, y_4\} = 1. \quad (53)$$

In this way, we have the following forms for the dynamical functions  $Q_i$  according to [18]

$$Q_1 = -y_4 = -e^{\frac{x_4}{2}}, \quad Q_2 = -\frac{y_3}{2} = -\frac{e^{-\frac{x_4}{2}}(-x_1 + e^{\frac{x_4}{2}}x_1 - x_2x_3)}{4x_2^2}, \quad (54)$$

$$Q_3 = 2y_1y_3 + y_2y_4 = \frac{x_1}{x_2}, \quad Q_4 = -y_2y_3 = \frac{e^{-3x_4/2}x_1^2}{2x_2^3} - \frac{e^{-x_4}x_1^2}{2x_2^3} + \frac{e^{-3x_4/2}x_1x_3}{x_2^2} - \frac{e^{-x_4}x_1x_3}{2x_2^2} + \frac{e^{-3x_4/2}x_3^2}{2x_2},$$

such that, they satisfy the following Poisson brackets by use of (52) or (53) as follows:

$$\{Q_1, Q_3\} = -Q_1, \quad \{Q_1, Q_4\} = 2Q_2, \quad \{Q_2, Q_3\} = -2Q_2, \quad \{Q_3, Q_4\} = Q_4, \quad (55)$$

i.e., a Poisson bracket  $\{Q_i, Q_j\} = f_{ij}^k Q_k$ , where,  $f_{ij}^k$  are the structure constants of the symmetry Lie algebra  $\mathbf{A}_{4,9}^{\mathbf{i},\mathbf{ii}}$ . The invariants of the above system are  $(Q_1, Q_2)$  or  $(Q_2, Q_4)$ , such that one can consider one of these  $Q_i$  as Hamiltonian of the integrable systems.

**Example 2)** In this example the role of phase space and symmetry group of example 1 are interchanged, i.e., the Lie group  $\mathbf{A}_{4,9}^{\mathbf{i},\mathbf{ii}}$  plays the role of phase space and  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$  plays the role of as symmetry group :  
For this example, the Darboux coordinates have the following forms [18]:

$$y_1 = x_1, \quad y_2 = -\frac{2e^{x_3}x_1x_4 + x_3}{x_1}, \quad y_3 = \frac{-x_2 + e^{-x_3}x_2 + 2x_1x_4}{x_1^2}, \quad y_4 = e^{-x_3}, \quad (56)$$

such that, we have the following forms for the symplectic structure (according table 6):

$$\{x_1, x_2\} = -x_1^2, \quad \{x_1, x_4\} = \frac{1}{2}e^{-x_3}x_1, \quad \{x_2, x_3\} = x_1, \quad \{x_2, x_4\} = e^{-x_3}x_2, \quad \{x_3, x_4\} = \frac{-1}{2}(1 - e^{-x_3}), \quad (57)$$

also, we have the following forms for the dynamical functions  $Q_i$  [18]

$$Q_1 = -y_3 = -\frac{-x_2 + e^{-x_3}x_2 + 2x_1x_4}{x_1^2}, \quad Q_2 = -y_4 = -e^{-x_3}, \quad (58)$$

$$Q_3 = -y_2y_3 = \left(\frac{2e^{x_3}x_1x_4 + x_3}{x_1}\right)\left(\frac{-x_2 + e^{-x_3}x_2 + 2x_1x_4}{x_1^2}\right), \quad Q_4 = -\frac{1}{2}y_1y_3 - y_2y_4 = \frac{x_2}{2x_1} + \frac{e^{-x_3}x_2}{x_1} + x_4,$$

with them satisfying the following Poisson brackets according to (57) as follows:

$$\{Q_1, Q_4\} = \frac{1}{2}Q_1, \quad \{Q_2, Q_3\} = Q_1, \quad \{Q_2, Q_4\} = Q_2, \quad \{Q_3, Q_4\} = -\frac{1}{2}Q_3, \quad (59)$$

i.e., a Poisson bracket  $\{Q_i, Q_j\} = f_{ij}^k Q_k$ , where,  $f_{ij}^k$  are the structure constants of the symmetry Lie algebra  $\mathbf{A}_{4,9}^{-\frac{1}{2}}$ . Where the invariants of the system are  $(Q_1, Q_2)$  or  $(Q_1, Q_3)$ , such that one can consider one of these  $Q_i$  as Hamiltonian of the integrable systems.

## 7 Conclusion

We classify all four dimensional real Lie bialgebras of symplectic type and obtain the classical r-matrices, Poisson and symplectic structures on all of the corresponding four dimensional Poisson-Lie groups. We also give two examples as the physical application, such that for these integrable systems the Poisson-Lie group  $G$  plays the role of phase space and its dual Lie group  $\tilde{G}$  plays the role of symmetry group of the system. Calculation of all such systems and also systems for which the role of  $G$  and  $\tilde{G}$  are replaced and the relation between such systems is under investigation.

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## Appendix A:

**Table 2:** Four dimensional real Lie bialgebras of symplectic type.

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Non-zero commutation relations of $\tilde{\mathfrak{g}}$	Comments
$A_{4,1}$	$4A_1$		
	$A_{4,1}.i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$A_{4,1}.ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3$	
	$A_{4,1}.iii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4$	
	$A_{4,9}^0.i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1 + \tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = q\tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^4$	$q \in \mathfrak{R} - \{0\}$
	$A_{4,9}^0.ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^4$	
	$A_{4,9}^0.iii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3 + \tilde{X}^4$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4$	
	$(VI_0 \oplus R).i$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3$	
	$(VI_0 \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^4$	
	$(VII_0 \oplus R).i$	$[\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3$	
$A_{4,2}^{-1}$	$4A_1$		
	$A_{4,2}^{-1}.i$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1 - \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$A_{4,5}^{-1,-1}.i$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).v$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).vi$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4$	
$A_{4,3}$	$4A_1$		
	$A_{4,3}.i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$	
	$A_{4,3}.ii$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4$	
	$A_{4,3}.iii$	$[\tilde{X}^1, \tilde{X}^4] = q\tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^4] = -q\tilde{X}^3$	$q \in \mathfrak{R} - \{0\}$
	$A_{4,3}.iv$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = q\tilde{X}^4$	$q \in \mathfrak{R} - \{0\}$
	$(A_2 \oplus A_2).i$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$(A_2 \oplus A_2).ii$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$	
	$(A_2 \oplus A_2).iii$	$[\tilde{X}^1, \tilde{X}^4] = q\tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = -q\tilde{X}^3$	$q \in \mathfrak{R} - \{0\}$
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).v$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).vi$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(III \oplus R).i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,$	
	$(III \oplus R).ii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3,$	
	$(VI_0 \oplus R).iii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$	
$A_{4,5}^{-1,-1}$	$4A_1$		
	$A_{4,5}^{-1,-1}.i$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$(II \oplus R).vii$	$[\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^4,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	

**Table 2:** (Continued.)

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Non-zero commutation relations of $\tilde{\mathfrak{g}}$	Comments
$A_{4,5}^{-1,b}$	$4A_1$		
	$A_{4,5}^{-1,b}.i$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^2, \quad [\tilde{X}^3, \tilde{X}^4] = b\tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
	$(II \oplus R).viii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
$A_{4,5}^{a,-1}$	$4A_1$		
	$A_{4,5}^{a,-1}.i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = a\tilde{X}^4$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).v$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).ix$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
$A_{4,5}^{a,-a}$	$4A_1$		
	$A_{4,5}^{a,-a}.i$	$[\tilde{X}^1, \tilde{X}^2] = -a\tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).viii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).v$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
$A_{4,6}^{a,0}$	$4A_1$		
	$A_{4,6}^{a,0}.i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^4] = -a\tilde{X}^4$	
	$(II \oplus R).iii$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
$A_{4,7}$	$4A_1$		
	$A_{4,7}.i$	$[\tilde{X}^1, \tilde{X}^2] = \frac{1}{2}(\tilde{X}^2 - \tilde{X}^3), \quad [\tilde{X}^1, \tilde{X}^3] = \frac{1}{2}\tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = 2\tilde{X}^4$	
	$A_{4,9}^{-\frac{1}{2}}.i$	$[\tilde{X}^1, \tilde{X}^2] = 2\tilde{X}^1, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
$A_{4,9}^0$	$4A_1$		
	$A_{4,9}^0.iv$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$A_{4,9}^0.v$	$[\tilde{X}^1, \tilde{X}^3] = q_1\tilde{X}^1 + \tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^4] = q_2\tilde{X}^2, \quad [\tilde{X}^2, \tilde{X}^3] = q_1\tilde{X}^2,$	$q_1, q_2 \in \Re - \{0\}$
	$(II \oplus R).x$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^2 + \tilde{X}^4,$	
	$(II \oplus R).viii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
	$(II \oplus R).xi$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^2 - \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^2 + \tilde{X}^4,$	
	$(VI_0 \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^2,$	
	$(VI_0 \oplus R).v$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^4,$	
	$(VII_0 \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^2,$	
$A_{4,9}^{-\frac{1}{2}}$	$4A_1$		
	$A_{4,9}^{-\frac{1}{2}}.ii$	$[\tilde{X}^1, \tilde{X}^2] = -2\tilde{X}^2, \quad [\tilde{X}^1, \tilde{X}^3] = 4\tilde{X}^3 + \tilde{X}^4, \quad [\tilde{X}^1, \tilde{X}^4] = 2\tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	

**Table 2:** (Continued.)

$\mathfrak{g}$	$\tilde{\mathfrak{g}}$	Non-zero commutation relations of $\tilde{\mathfrak{g}}$	Comments
$A_{4,9}^1$	$A_{4,9}^{-\frac{1}{2}.iii}$	$[\tilde{X}^1, \tilde{X}^2] = -2\tilde{X}^2, [\tilde{X}^1, \tilde{X}^3] = 4\tilde{X}^3, [\tilde{X}^1, \tilde{X}^4] = 2\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4$	
	$A_{4,9}^{-\frac{1}{2}.iv}$	$[\tilde{X}^1, \tilde{X}^2] = \frac{1}{2}\tilde{X}^2, [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = \frac{1}{2}\tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4$	
	$A_{4,9}^1.ii$	$[\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^1, [\tilde{X}^1, \tilde{X}^4] = 2\tilde{X}^2, [\tilde{X}^2, \tilde{X}^3] = -2\tilde{X}^2, [\tilde{X}^3, \tilde{X}^4] = \tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
	$(III \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
	$4A_1$		
	$A_{4,9}^1.i$	$[\tilde{X}^1, \tilde{X}^2] = -\frac{1}{4}\tilde{X}^2, [\tilde{X}^1, \tilde{X}^3] = -\frac{1}{4}\tilde{X}^3, [\tilde{X}^1, \tilde{X}^4] = -\frac{1}{2}\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^4$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
	$4A_1$		
	$A_{4,9}^b$		
$A_{4,9}^b$	$4A_1$		
	$A_{4,9}^b.i$	$[\tilde{X}^1, \tilde{X}^2] = b\tilde{X}^2, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^3, [\tilde{X}^1, \tilde{X}^4] = (1+b)\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = (1+b)^2\tilde{X}^4$	
	$(II \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^4,$	
$A_{4,11}^b$	$4A_1$		
	$A_{4,11}^b.i$	$[\tilde{X}^1, \tilde{X}^2] = \frac{1}{2}\tilde{X}^2 - \frac{1}{2b}\tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = \frac{1}{2b}\tilde{X}^2 + \frac{1}{2}\tilde{X}^3, [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = 2b\tilde{X}^4$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	
$A_{4,12}$	$4A_1$		
	$A_{4,12}.i$	$[\tilde{X}^1, \tilde{X}^3] = -q_1\tilde{X}^1, [\tilde{X}^1, \tilde{X}^4] = q_2\tilde{X}^1 + q_1\tilde{X}^2, [\tilde{X}^2, \tilde{X}^3] = -q_1\tilde{X}^2,$ $[\tilde{X}^2, \tilde{X}^4] = -q_1\tilde{X}^1 + q_2\tilde{X}^2$	$q_1, q_2 \in \mathfrak{R} - \{0\}$
	$A_{4,12}.ii$	$[\tilde{X}^1, \tilde{X}^2] = q\tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^4, [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^4$	$q \in \mathfrak{R} - \{0\}$
	$A_{4,12}.iii$	$[\tilde{X}^1, \tilde{X}^3] = -q\tilde{X}^1, [\tilde{X}^1, \tilde{X}^4] = q\tilde{X}^2, [\tilde{X}^2, \tilde{X}^3] = -q\tilde{X}^2, [\tilde{X}^2, \tilde{X}^4] = -q\tilde{X}^1$	$q \in \mathfrak{R} - \{0\}$
	$(II \oplus R).xii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,$	
	$(VII_0 \oplus R).iii$	$[\tilde{X}^1, \tilde{X}^2] = -q(\tilde{X}^1 + \tilde{X}^3), [\tilde{X}^1, \tilde{X}^3] = q(\tilde{X}^2 - \tilde{X}^4), [\tilde{X}^2, \tilde{X}^3] = -q(\tilde{X}^1 + \tilde{X}^3)$	$q \in \mathfrak{R} - \{0\}$
$A_2 \oplus A_2$	$4A_1$		
	$(A_2 \oplus A_2).iv$	$[\tilde{X}^1, \tilde{X}^2] = q_1\tilde{X}^2, [\tilde{X}^1, \tilde{X}^4] = q_2\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = q_2\tilde{X}^2,$	$q_1, q_2 \in \mathfrak{R} - \{0\}$
	$(A_2 \oplus A_2).v$	$[\tilde{X}^1, \tilde{X}^4] = q\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = q\tilde{X}^2,$	$q \in \mathfrak{R} - \{0\}$
	$(A_2 \oplus A_2).vi$	$[\tilde{X}^1, \tilde{X}^2] = q\tilde{X}^2, [\tilde{X}^3, \tilde{X}^4] = \tilde{X}^3,$	$q \in \mathfrak{R} - \{0\}$
	$(A_2 \oplus A_2).vii$	$[\tilde{X}^1, \tilde{X}^4] = q_1\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = q_1\tilde{X}^2, [\tilde{X}^3, \tilde{X}^4] = q_2\tilde{X}^4,$	$q_1, q_2 \in \mathfrak{R} - \{0\}$
	$(II \oplus R).xiii$	$[\tilde{X}^2, \tilde{X}^4] = \tilde{X}^1 + \tilde{X}^3,$	
	$(III \oplus R).i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1,$	
	$(III \oplus R).iii$	$[\tilde{X}^3, \tilde{X}^4] = \tilde{X}^3,$	
	$(III \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1 + q\tilde{X}^2,$	$q \in \mathfrak{R} - \{0\}$
	$(III \oplus R).v$	$[\tilde{X}^3, \tilde{X}^4] = \tilde{X}^3 + q\tilde{X}^4,$	$q \in \mathfrak{R} - \{0\}$
	$(VI_0 \oplus R).vi$	$[\tilde{X}^1, \tilde{X}^2] = -q\tilde{X}^2, [\tilde{X}^1, \tilde{X}^4] = q\tilde{X}^4, [\tilde{X}^2, \tilde{X}^3] = q\tilde{X}^2, [\tilde{X}^3, \tilde{X}^4] = q\tilde{X}^4,$	$q \in \mathfrak{R} - \{0\}$
	$(VI_0 \oplus R).vii$	$[\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3, [\tilde{X}^3, \tilde{X}^4] = -\tilde{X}^3,$	
	$(VI_0 \oplus R).viii$	$[\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^2, [\tilde{X}^3, \tilde{X}^4] = -\tilde{X}^3,$	
	$4A_1$		
	$(VI_0 \oplus R).ix$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^2, [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^1,$	
	$(II \oplus R).xiv$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3,$	
	$(II \oplus R).xii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,$	
$VI_0 \oplus R$	$(II \oplus R).i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3 + \tilde{X}^4,$	
	$(III \oplus R).vi$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3,$	
	$(III \oplus R).vii$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^1 - \tilde{X}^2 + \tilde{X}^3, [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^1 + \tilde{X}^2 - \tilde{X}^3,$	

**Table 2:** (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero commutation relations of $\tilde{\mathbf{g}}$	Comments
$VII_0 \oplus R$	$4R$		
	$(VII_0 \oplus R).iv$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^2, \quad [\tilde{X}^2, \tilde{X}^4] = -\tilde{X}^1,$	
	$(II \oplus R).xiv$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3,$	
	$(II \oplus R).xii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3,$	
	$(II \oplus R).i$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3 + \tilde{X}^4,$	
$III \oplus R$	$4A_1$		
	$(III \oplus R).xiii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, \quad [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1,$	
	$(III \oplus R).vix$	$[\tilde{X}^2, \tilde{X}^4] = \tilde{X}^4, \quad [\tilde{X}^3, \tilde{X}^4] = -\tilde{X}^4,$	
	$(II \oplus R).xv$	$[\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^3, \tilde{X}^4] = \tilde{X}^1,$	
$II \oplus R$	$4A_1$		
	$(II \oplus R).xiv$	$[\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3,$	
	$(II \oplus R).xvi$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = \tilde{X}^3,$	
	$(II \oplus R).xviii$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3, \quad [\tilde{X}^1, \tilde{X}^4] = -\tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \tilde{X}^3,$	
	$(II \oplus R).xviii$	$[\tilde{X}^1, \tilde{X}^2] = -\tilde{X}^3,$	
	$(II \oplus R).ii$	$[\tilde{X}^1, \tilde{X}^2] = \tilde{X}^4,$	

## Appendix B:

**Table 3:** Four dimensional coboundary and bi-r-matrix Lie bialgebras of symplectic type.

$(\mathbf{g}, \tilde{\mathbf{g}})$	$r$	$[[r, r]]$	$\tilde{r}$	$[[\tilde{r}, \tilde{r}]]$
$(A_{4,1}, A_{4,1}.iii)$	$aX_1 \otimes X_1 + c(X_1 \wedge X_3 - X_2 \otimes X_2) +$ $bX_1 \wedge X_2 + X_1 \wedge X_4 - X_2 \wedge X_3$	0	$d(\tilde{X}^2 \otimes \tilde{X}^4 + \tilde{X}^4 \otimes \tilde{X}^2 - \tilde{X}^3 \otimes \tilde{X}^3)$ $+ \tilde{X}^2 \wedge \tilde{X}^3 + e\tilde{X}^3 \wedge \tilde{X}^4 + \tilde{X}^4 \wedge \tilde{X}^1$ $+ f\tilde{X}^4 \otimes \tilde{X}^4$	0
$(A_{4,2}^{-1}, A_{4,2}^{-1}.i)$	$aX_1 \otimes X_2 + X_1 \wedge X_3 + bX_2 \otimes X_1 -$ $X_2 \wedge X_4$	0	$-\tilde{X}^1 \wedge \tilde{X}^3 + \tilde{X}^2 \wedge \tilde{X}^4 + c\tilde{X}^3 \otimes \tilde{X}^4$ $+ d\tilde{X}^4 \otimes \tilde{X}^3$	0
$(A_{4,3}, A_{4,3}.ii)$	$-X_1 \wedge X_4 + aX_2 \otimes X_2 + bX_2 \wedge X_3$	0	$\tilde{X}^1 \wedge \tilde{X}^4 + c\tilde{X}^2 \wedge \tilde{X}^3 + d\tilde{X}^3 \otimes \tilde{X}^3$	0
$(A_{4,5}^{-1,-1}, A_{4,5}^{-1,-1}.i)$	$aX_1 \otimes X_2 + bX_1 \otimes X_3 + cX_2 \otimes X_1 +$ $X_2 \wedge X_4 + dX_3 \otimes X_1$	0	$e\tilde{X}^1 \otimes \tilde{X}^3 - \tilde{X}^2 \wedge \tilde{X}^4 + f\tilde{X}^3 \otimes \tilde{X}^1 +$ $k\tilde{X}^3 \otimes \tilde{X}^4 + h\tilde{X}^4 \otimes \tilde{X}^3$	0
$(A_{4,5}^{-1,b}, A_{4,5}^{-1,b}.i)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 - X_3 \wedge X_4$	0	$c\tilde{X}^1 \otimes \tilde{X}^2 + d\tilde{X}^2 \otimes \tilde{X}^1 + \tilde{X}^3 \wedge \tilde{X}^4$	0
$(A_{4,5}^{a,-1}, A_{4,5}^{a,-1}.i)$	$cX_1 \otimes X_3 - X_2 \wedge X_4 + dX_3 \otimes X_1$	0	$e\tilde{X}^1 \otimes \tilde{X}^3 + \tilde{X}^2 \wedge \tilde{X}^4 + f\tilde{X}^3 \otimes \tilde{X}^1$	0
$(A_{4,5}^{a,-a}, A_{4,5}^{a,-a}.i)$	$-X_1 \wedge X_4 + cX_2 \otimes X_3 + dX_3 \otimes X_2$	0	$\tilde{X}^1 \wedge \tilde{X}^4 + e\tilde{X}^2 \otimes \tilde{X}^3 + f\tilde{X}^3 \otimes \tilde{X}^2$	0
$(A_{4,6}^{a,0}, A_{4,6}^{a,0}.i)$	$X_1 \wedge X_4 + c(X_2 \otimes X_2 + X_3 \otimes X_3) +$ $eX_2 \wedge X_3$	0	$-\tilde{X}^1 \wedge \tilde{X}^4 + d(\tilde{X}^2 \otimes \tilde{X}^2 + \tilde{X}^3 \otimes \tilde{X}^3) -$ $f\tilde{X}^2 \wedge \tilde{X}^3$	0
$(A_{4,7}, A_{4,7}.i)$	$-\frac{1}{2}X_1 \wedge X_4 - X_2 \wedge X_3$	0	$2\tilde{X}^1 \wedge \tilde{X}^4 + \tilde{X}^2 \wedge \tilde{X}^3$	0
$(A_{4,9}^0, A_{4,9}^0.iv)$	$-\frac{1}{2}X_1 \wedge X_2 - X_1 \wedge X_4 - X_2 \wedge X_3$	0	$\tilde{X}^1 \wedge \tilde{X}^4 + \tilde{X}^2 \wedge \tilde{X}^3 + \frac{1}{2}\tilde{X}^3 \wedge \tilde{X}^4$	0
$(A_{4,9}^0, VI_0 \oplus R.v)$	$X_1 \wedge X_4$	0	$-\tilde{X}^1 \wedge \tilde{X}^4 + \tilde{X}^2 \wedge \tilde{X}^3 + c\tilde{X}^2 \otimes \tilde{X}^4 +$ $d\tilde{X}^4 \otimes \tilde{X}^2 + e\tilde{X}^3 \otimes \tilde{X}^3$	$\tilde{X}^2 \wedge \tilde{X}^3 \wedge \tilde{X}^4$
$(A_{4,9}^{-\frac{1}{2}}, A_{4,9}^{-\frac{1}{2}}.iii)$	$aX_1 \wedge X_3 - 4X_1 \wedge X_4 - 2X_2 \wedge X_3$	0	$\frac{1}{4}\tilde{X}^1 \wedge \tilde{X}^4 + \frac{1}{2}\tilde{X}^2 \wedge \tilde{X}^3 + c\tilde{X}^2 \wedge \tilde{X}^4$	0
$(A_{4,9}^{-\frac{1}{2}}, A_{4,9}^{-\frac{1}{2}}.iv)$	$aX_1 \wedge X_3 - X_2 \wedge X_4$	0	$b\tilde{X}^1 \wedge \tilde{X}^3 + \tilde{X}^2 \wedge \tilde{X}^4$	0

**Table 3:** (Continued.)

$(\mathbf{g}, \tilde{\mathbf{g}})$	$r$	$[[r, r]]$	$\tilde{r}$	$[[\tilde{r}, \tilde{r}]]$
$(A_{4,9}^{-\frac{1}{2}}, A_{4,9}^1.ii)$	$aX_1 \wedge X_3 + 2X_3 \wedge X_4$	0	$-\frac{1}{2}\tilde{X}^3 \wedge \tilde{X}^4$	0
$(A_{4,9}^1, A_{4,9}^1.i)$	$\frac{1}{4}X_1 \wedge X_4 + \frac{1}{2}X_2 \wedge X_3$	0	$-4\tilde{X}^1 \wedge \tilde{X}^4 - 2\tilde{X}^2 \wedge \tilde{X}^3$	0
$(A_{4,9}^b, A_{4,9}^b.i)$	$-X_1 \wedge X_4 - (1+b)X_2 \wedge X_3$	0	$\tilde{X}^1 \wedge \tilde{X}^4 + \frac{1}{1+b}\tilde{X}^2 \wedge \tilde{X}^3$	0
$(A_{4,11}^b, A_{4,11}^b.i)$	$-\frac{1}{2b}X_1 \wedge X_4 - X_2 \wedge X_3$	0	$2b\tilde{X}^1 \wedge \tilde{X}^4 + \tilde{X}^2 \wedge \tilde{X}^3$	0
$(A_{4,12}, A_{4,12}.ii)$	$-\frac{q}{2}X_1 \wedge X_2 - X_1 \wedge X_4 + X_2 \wedge X_3$	0	$\tilde{X}^1 \wedge \tilde{X}^4 - \tilde{X}^2 \wedge \tilde{X}^3 - \frac{q}{2}\tilde{X}^3 \wedge \tilde{X}^4$	0
$(A_2 \oplus A_2, (A_2 \oplus A_2).v)$	$-qX_1 \wedge X_3$	0	$\frac{1}{q}\tilde{X}^1 \wedge \tilde{X}^3$	0
$(VI_0 \oplus R, (VI_0 \oplus R).ix)$	$a(X_1 \otimes X_1 - X_2 \otimes X_2) + bX_1 \wedge X_2 +$ $X_3 \wedge X_4 + cX_4 \otimes X_4$	0	$d(\tilde{X}^1 \otimes \tilde{X}^1 - \tilde{X}^2 \otimes \tilde{X}^2) + e\tilde{X}^1 \wedge \tilde{X}^2 +$ $f\tilde{X}^3 \otimes \tilde{X}^3 - \tilde{X}^3 \wedge \tilde{X}^4$	0
$(VII_0 \oplus R, (VII_0 \oplus R).iv)$	$a(X_1 \otimes X_1 + X_2 \otimes X_2) + bX_1 \wedge X_2 +$ $X_3 \wedge X_4 + cX_4 \otimes X_4$	0	$d(\tilde{X}^1 \otimes \tilde{X}^1 + \tilde{X}^2 \otimes \tilde{X}^2) + e\tilde{X}^1 \wedge \tilde{X}^2 +$ $f\tilde{X}^3 \otimes \tilde{X}^3 - \tilde{X}^3 \wedge \tilde{X}^4$	0
$(III \oplus R, (III \oplus R).xiii)$	$-\frac{1}{2}(X_1 \wedge X_2 + X_1 \wedge X_3 + X_2 \wedge X_3) +$ $b(X_2 \otimes X_2 - X_2 \otimes X_3 - X_3 \otimes X_2 +$ $X_3 \otimes X_3) + c(X_2 \otimes X_4 - X_3 \otimes X_4)$ $+d(X_4 \otimes X_2 - X_4 \otimes X_3) + e(X_4 \otimes X_4)$	0	$\tilde{X}^1 \wedge \tilde{X}^2 + \tilde{X}^1 \wedge \tilde{X}^3 + i\tilde{X}^4 \otimes \tilde{X}^4 +$ $f(\tilde{X}^1 \otimes \tilde{X}^4 - \tilde{X}^2 \otimes \tilde{X}^4 + \tilde{X}^3 \otimes \tilde{X}^4) +$ $g(\tilde{X}^4 \otimes \tilde{X}^1 - \tilde{X}^4 \otimes \tilde{X}^2 + \tilde{X}^4 \otimes \tilde{X}^3)$	0
$(II \oplus R, (II \oplus R).xiv)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + cX_1 \wedge X_3 +$ $dX_1 \wedge X_4 - X_2 \wedge X_4 + eX_4 \otimes X_4$	0	$f\tilde{X}^1 \wedge \tilde{X}^3 + g\tilde{X}^2 \otimes \tilde{X}^2 + h\tilde{X}^2 \wedge \tilde{X}^3 +$ $\tilde{X}^2 \wedge \tilde{X}^4 - j\tilde{X}^3 \otimes \tilde{X}^3 + k\tilde{X}^3 \wedge \tilde{X}^4$	0

**Table 4:** Four dimensional coboundary Lie bialgebras of symplectic type.

$(\mathbf{g}, \tilde{\mathbf{g}})$	$\mathbf{r}$	$[[\mathbf{r}, \mathbf{r}]]$
$(A_{4,1}, 4A_1)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + c(X_1 \wedge X_3 - X_2 \otimes X_2)$	0
$(A_{4,1}, (II \oplus R).i)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + cX_1 \otimes X_3 + X_1 \wedge X_4 - (1-c)X_2 \otimes X_2 + (2+c)X_3 \otimes X_1$	0
$(A_{4,1}, (II \oplus R).ii)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + cX_1 \otimes X_3 - (1+c)X_2 \otimes X_2 + (2+c)X_3 \otimes X_1$	0
$(A_{4,1}, (II \oplus R).iv)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + c(X_1 \wedge X_3 - X_2 \otimes X_2) - X_2 \wedge X_3$	0
$(A_{4,2}^{-1}, 4A_1)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1$	0
$(A_{4,2}^{-1}, A_{4,5}^{-1,-1}.i)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 - X_2 \wedge X_4$	0
$(A_{4,2}^{-1}, (II \oplus R).iii)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 - \frac{1}{2}X_2 \wedge X_3$	0
$(A_{4,2}^{-1}, (II \oplus R).v)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 - X_1 \wedge X_3 - \frac{1}{2}X_2 \wedge X_3$	0
$(A_{4,2}^{-1}, (II \oplus R).ii)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 - X_1 \wedge X_3$	0
$(A_{4,3}, 4A_1)$	$aX_2 \otimes X_2 + bX_2 \wedge X_3$	0
$(A_{4,3}, (II \oplus R).iv)$	$X_1 \wedge X_2 - X_1 \wedge X_3 + aX_2 \otimes X_2 + bX_2 \wedge X_3$	0
$(A_{4,3}, (III \oplus R).i)$	$aX_2 \otimes X_2 + bX_2 \wedge X_3 - X_2 \wedge X_4$	0
$(A_{4,5}^{-1,-1}, 4A_1)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 + cX_1 \otimes X_3 + dX_3 \otimes X_1$	0
$(A_{4,5}^{-1,-1}, (II \oplus R).vii)$	$aX_1 \otimes X_2 + bX_2 \otimes X_1 + cX_1 \otimes X_3 + dX_3 \otimes X_1 - \frac{1}{2}X_2 \wedge X_3$	0
$(A_{4,5}^{-1,b}, 4A_1)$	$cX_1 \otimes X_2 + dX_2 \otimes X_1$	0
$(A_{4,5}^{-1,b}, (II \oplus R).iv)$	$cX_1 \otimes X_2 + dX_2 \otimes X_1 - \frac{1}{1+b}X_1 \wedge X_3$	0
$(A_{4,5}^{-1,b}, (II \oplus R).iii)$	$cX_1 \otimes X_2 + dX_2 \otimes X_1 + \frac{1}{1-b}X_2 \wedge X_3$	0

**Table 4:** (Continued.)

$(\mathbf{g}, \tilde{\mathbf{g}})$	$\mathbf{r}$	$[[\mathbf{r}, \mathbf{r}]]$
$(A_{4,5}^{a,-1}, 4A_1)$	$cX_1 \otimes X_3 + dX_3 \otimes X_1$	0
$(A_{4,5}^{a,-1}, (II \oplus R).iii)$	$cX_1 \otimes X_3 + dX_3 \otimes X_1 + \frac{1}{1-a}X_2 \wedge X_3$	0
$(A_{4,5}^{a,-1}, (II \oplus R).v)$	$-\frac{1}{1+a}X_1 \wedge X_2 + cX_1 \otimes X_3 + dX_3 \otimes X_1 - \frac{1}{a-1}X_2 \wedge X_3$	0
$(A_{4,5}^{a,-a}, 4A_1)$	$cX_2 \otimes X_3 + dX_3 \otimes X_2$	0
$(A_{4,5}^{a,-a}, (II \oplus R).iv)$	$-\frac{1}{1-a}X_1 \wedge X_3 + cX_2 \otimes X_3 + dX_3 \otimes X_2$	0
$(A_{4,5}^{a,-a}, (II \oplus R).viii)$	$-\frac{1}{1+a}X_1 \wedge X_2 - \frac{1}{1-a}X_1 \wedge X_3 + cX_2 \otimes X_3 + dX_3 \otimes X_2$	0
$(A_{4,6}^{a,0}, 4A_1)$	$c(X_2 \otimes X_2 + X_3 \otimes X_3) + dX_2 \wedge X_3$	0
$(A_{4,6}^{a,0}, (II \oplus R).ii)$	$\frac{-a}{1+a^2}X_1 \wedge X_2 - \frac{1}{1+a^2}X_1 \wedge X_3 + c(X_2 \otimes X_2 + X_3 \otimes X_3) + dX_2 \wedge X_3$	0
$(A_{4,7}, (A_{4,9}^{-\frac{1}{2}}).i)$	$-X_2 \wedge X_4$	0
$(A_{4,7}, (II \oplus R).iv)$	$\frac{1}{9}X_1 \wedge X_2 - \frac{1}{3}X_1 \wedge X_3$	0
$(A_{4,7}, (II \oplus R).ii)$	$-\frac{1}{3}X_1 \wedge X_2$	0
$(A_{4,9}^0, (II \oplus R).viii))$	$-\frac{1}{2}X_1 \wedge X_2 - X_1 \wedge X_3$	0
$(A_{4,9}^0, (II \oplus R).iv))$	$-X_1 \wedge X_3$	0
$(A_{4,9}^0, (II \oplus R).ii)$	$-\frac{1}{2}X_1 \wedge X_2$	0
$(A_{4,9}^{-\frac{1}{2}}, 4A_1)$	$cX_1 \wedge X_3$	0
$(A_{4,9}^{-\frac{1}{2}}, (II \oplus R).ii)$	$-\frac{2}{3}X_1 \wedge X_2 + cX_1 \wedge X_3$	0
$(A_{4,9}^1, (II \oplus R).ii)$	$-\frac{1}{3}X_1 \wedge X_2$	0
$(A_{4,9}^b, (II \oplus R).iv)$	$-\frac{1}{1+2b}X_2 \wedge X_3$	0
$(A_{4,11}^b, (II \oplus R).ii)$	$-\frac{3b}{1+9b^2}X_1 \wedge X_2 - \frac{1}{1+9b^2}X_1 \wedge X_3$	0
$(A_{4,12}, (II \oplus R).xii)$	$-\frac{1}{2}X_1 \wedge X_2$	0
$((A_2 \oplus A_2), (II \oplus R).xiii)$	$X_2 \wedge X_4$	0
$((A_2 \oplus A_2), (III \oplus R).i)$	$X_1 \wedge X_2$	0
$((A_2 \oplus A_2), (III \oplus R).iii)$	$X_3 \wedge X_4$	0
$((A_2 \oplus A_2), (VI_0 \oplus R).vii)$	$-X_1 \wedge X_4 + X_2 \wedge X_4 - X_3 \wedge X_4$	0
$((A_2 \oplus A_2), (VI_0 \oplus R).viii)$	$-X_1 \wedge X_4 - X_3 \wedge X_4$	0
$(VI_0 \oplus R, 4A_1)$	$a(X_1 \otimes X_1 - X_2 \otimes X_2) + bX_1 \wedge X_2 + cX_4 \otimes X_4$	0
$(VI_0 \oplus R, (II \oplus R).xiv)$	$a(X_1 \otimes X_1 - X_2 \otimes X_2) + bX_1 \wedge X_2 - X_2 \wedge X_4 + cX_4 \otimes X_4$	0
$(VII_0 \oplus R, 4A_1)$	$a(X_1 \otimes X_1 + X_2 \otimes X_2) + bX_1 \wedge X_2 + cX_4 \otimes X_4$	0
$(VII_0 \oplus R, (II \oplus R).xiv)$	$a(X_1 \otimes X_1 + X_2 \otimes X_2) + bX_1 \wedge X_2 - X_2 \wedge X_4 + cX_4 \otimes X_4$	0
$(III \oplus R, 4A_1)$	$a(X_2 \otimes X_2 + X_3 \otimes X_3 - X_2 \otimes X_3 - X_3 \otimes X_2) + b(X_2 \otimes X_4 - X_3 \otimes X_4) + d(X_4 \otimes X_2 - X_4 \otimes X_3) + cX_4 \otimes X_4$	0
$(II \oplus R, 4A_1)$	$aX_1 \otimes X_1 + bX_1 \wedge X_2 + cX_1 \wedge X_3 + dX_1 \wedge X_4 + eX_4 \otimes X_4$	0
$((VI_0 \oplus R).iv, A_{4,9}^0)$	$-X_1 \wedge X_2 + a(X_2 \otimes X_2 - X_4 \otimes X_4) + bX_3 \otimes X_3 + dX_2 \wedge X_4 + X_3 \wedge X_4$	$-X_1 \wedge X_2 \wedge X_4$
$((VII_0 \oplus R).ii, A_{4,9}^0)$	$-X_1 \wedge X_2 + a(X_2 \otimes X_2 + X_4 \otimes X_4) + bX_3 \otimes X_3 + dX_2 \wedge X_4 - X_3 \wedge X_4$	$-X_1 \wedge X_2 \wedge X_4$
$((A_2 \oplus A_2).i, A_{4,3})$	$X_3 \wedge X_4$	0
$((A_2 \oplus A_2).ii, A_{4,3})$	$X_1 \wedge X_4$	0



### Appendix C:

**Table 5:** Left and right invariant vector fields over four dimensional real Lie groups of symplectic type.

$\mathfrak{g}$	$\begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \\ X_4^L \end{pmatrix}$	$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \\ X_4^R \end{pmatrix}$
$A_{4,1}$	$\begin{pmatrix} \partial_1 \\ \partial_2 - x_4 \partial_1 \\ \partial_3 - x_4 \partial_2 + \frac{x_4^2}{2} \partial_1 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ \partial_4 - x_2 \partial_1 - x_3 \partial_2 \end{pmatrix}$
$A_{4,1}.iii$	$\begin{pmatrix} \partial_1 - x_3 \partial_4 - x_2 \partial_3 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 - x_1 \partial_3 + \frac{x_1^2}{2} \partial_4 \\ \partial_3 - x_1 \partial_4 \\ \partial_4 \end{pmatrix}$
$A_{4,2}^{-1}$	$\begin{pmatrix} e^{x_4} \partial_1 \\ e^{-x_4} \partial_2 \\ e^{-x_4} (\partial_3 - x_4 \partial_2) \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ x_1 \partial_1 - (x_2 + x_3) \partial_2 - x_3 \partial_3 + \partial_4 \end{pmatrix}$
$A_{4,2}^{-1}.i$	$\begin{pmatrix} e^{x_2} \partial_1 + x_2 \partial_4 \\ \partial_2 - x_4 \partial_4 + x_3 \partial_3 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 + x_1 \partial_1 + x_1 e^{-x_2} \partial \partial_4 \\ e^{x_2} \partial_3 \\ e^{-x_2} \partial_4 \end{pmatrix}$
$A_{4,3}$	$\begin{pmatrix} e^{-x_4} \partial_1 \\ \partial_2 \\ \partial_3 - x_4 \partial_2 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ \partial_4 - x_1 \partial_1 - x_3 \partial_2 \end{pmatrix}$
$A_{4,3}.ii$	$\begin{pmatrix} \partial_1 + x_2 \partial_3 - x_4 \partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 + x_1 \partial_3 \\ \partial_3 \\ e^{-x_1} \partial_4 \end{pmatrix}$
$A_{4,5}^{-1,-1}$	$\begin{pmatrix} e^{-x_4} \partial_1 \\ e^{x_4} \partial_2 \\ e^{x_4} \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + \partial_4 \end{pmatrix}$
$A_{4,5}^{-1,-1}.i$	$\begin{pmatrix} e^{x_2} \partial_1 \\ \partial_2 + x_3 \partial_3 - x_4 \partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ x_1 \partial_1 + \partial_2 \\ e^{x_2} \partial_3 \\ e^{-x_2} \partial_4 \end{pmatrix}$
$A_{4,5}^{-1,b}$	$\begin{pmatrix} e^{-x_4} \partial_1 \\ e^{x_4} \partial_2 \\ e^{-bx_4} \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_1 \partial_1 + x_2 \partial_2 - bx_3 \partial_3 + \partial_4 \end{pmatrix}$
$A_{4,5}^{-1,b}.i$	$\begin{pmatrix} e^{-x_3} \partial_1 \\ e^{x_3} \partial_2 \\ \partial_3 - bx_4 \partial_4 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_1 \partial_1 + x_2 \partial_2 + \partial_3 \\ e^{-bx_3} \partial_4 \end{pmatrix}$
$A_{4,5}^{a,-1}$	$\begin{pmatrix} e^{-x_4} \partial_1 \\ e^{-ax_4} \partial_2 \\ e^{x_4} \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_1 \partial_1 - ax_2 \partial_2 + x_3 \partial_3 + \partial_4 \end{pmatrix}$
$A_{4,5}^{a,-1}.i$	$\begin{pmatrix} e^{-x_2} \partial_1 \\ \partial_2 - x_3 \partial_3 - ax_4 \partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1 \partial_1 + \partial_2 \\ e^{-x_2} \partial_3 \\ e^{-ax_2} \partial_4 \end{pmatrix}$
$A_{4,5}^{a,-a}$	$\begin{pmatrix} e^{-x_4} \partial_1 \\ e^{-ax_4} \partial_2 \\ e^{ax_4} \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_1 \partial_1 - ax_2 \partial_2 + ax_3 \partial_3 + \partial_4 \end{pmatrix}$
$A_{4,5}^{a,-a}.i$	$\begin{pmatrix} \partial_1 + ax_2 \partial_2 - ax_3 \partial_3 - x_4 \partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{ax_1} \partial_2 \\ e^{-ax_1} \partial_3 \\ e^{-x_1} \partial_4 \end{pmatrix}$
$A_{4,7}$	$\begin{pmatrix} e^{-2x_4} \partial_1 \\ e^{-x_4} (\partial_2 - x_3 \partial_1) \\ e^{-x_4} (x_3 x_4 \partial_1 - x_4 \partial_2 + \partial_3) \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2 \partial_1 + \partial_3 \\ (-2x_1 + \frac{x_3^2}{2}) \partial_1 - (x_2 + x_3) \partial_2 - x_3 \partial_3 + \partial_4 \end{pmatrix}$

**Table 5:** (Continued.)

$A_{4,7}.i$	$\begin{pmatrix} \partial_1 - \frac{x_2}{2}\partial_2 + \frac{x_2-x_3}{2}\partial_3 - \frac{x_2^2+2x_4}{2}\partial_4 \\ \partial_2 - 2x_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{-\frac{x_1}{2}}(\partial_2 + \frac{x_1}{2}\partial_3 - x_1x_2\partial_4) \\ e^{-\frac{x_1}{2}}(\partial_3 - 2x_2\partial_4) \\ e^{-x_1}\partial_4 \end{pmatrix}$
$A_{4,9}^0$	$\begin{pmatrix} e^{-x_4}\partial_1 \\ e^{-x_4}(\partial_2 - x_3\partial_1) \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 - x_2\partial_1 \\ -x_1\partial_1 - x_2\partial_2 + \partial_4 \end{pmatrix}$
$A_{4,9}^0.iv$	$\begin{pmatrix} \partial_1 - x_3\partial_3 - (x_2 + x_4)\partial_4 \\ \partial_2 - x_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 + (e^{-x_1} - 1)\partial_4 \\ e^{-x_1}(\partial_3 - x_2\partial_4) \\ e^{-x_1}\partial_4 \end{pmatrix}$
$A_{4,9}^{-1/2}$	$\begin{pmatrix} e^{-\frac{x_4}{2}}\partial_1 \\ e^{-x_4}(\partial_2 - x_3\partial_1) \\ e^{\frac{x_4}{2}}\partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 - x_2\partial_1 \\ -\frac{x_1}{2}\partial_1 - x_2\partial_2 + \frac{x_3}{2}\partial_3 + \partial_4 \end{pmatrix}$
$A_{4,9}^{-1/2}.iii$	$\begin{pmatrix} \partial_1 + 2x_2\partial_2 - 4x_3\partial_3 - 2x_4\partial_4 \\ \partial_2 - x_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{2x_1}\partial_2 \\ e^{-4x_1}(\partial_3 - x_2\partial_4) \\ e^{-2x_1}\partial_4 \end{pmatrix}$
$A_{4,9}^{-1/2}.iv$	$\begin{pmatrix} e^{-\frac{x_2}{2}}\partial_1 - x_4\partial_3 \\ \partial_2 - \frac{x_3}{2}\partial_3 - x_4\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -\frac{x_1}{2}\partial_1 + \partial_2 \\ e^{-\frac{x_2}{2}}\partial_3 \\ -e^{-\frac{x_2}{2}}x_1\partial_3 + e^{-x_2}\partial_4 \end{pmatrix}$
$A_{4,9}^1$	$\begin{pmatrix} e^{-2x_4}\partial_1 \\ e^{-x_4}(\partial_2 - x_3\partial_1) \\ e^{-x_4}\partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 - x_2\partial_1 \\ -2x_1\partial_1 - x_2\partial_2 - x_3\partial_3 + \partial_4 \end{pmatrix}$
$A_{4,9}^1.i$	$\begin{pmatrix} \partial_1 + \frac{x_2}{4}\partial_2 + \frac{x_3}{4}\partial_3 + \frac{x_4}{4}\partial_4 \\ \partial_2 + x_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{\frac{x_1}{4}}\partial_2 \\ e^{\frac{x_1}{4}}(\partial_3 + x_2\partial_4) \\ e^{\frac{x_1}{2}}\partial_4 \end{pmatrix}$
$II \oplus R$	$\begin{pmatrix} \partial_1 \\ \partial_2 - x_3\partial_1 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 - x_2\partial_1 \\ \partial_4 \end{pmatrix}$
$(II \oplus R).xiv$	$\begin{pmatrix} \partial_1 - x_4\partial_3 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_1\partial_3 + \partial_4 \end{pmatrix}$
$III \oplus R$	$\begin{pmatrix} \partial_1 + (x_2 + x_3)(\partial_2 + \partial_3) \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{x_1}(\cosh(x_1)\partial_2 + \sinh(x_1)\partial_3) \\ e^{x_1}(\cosh(x_1)\partial_3 + \sinh(x_1)\partial_2) \\ \partial_4 \end{pmatrix}$
$(III \oplus R).xiii$	$\begin{pmatrix} e^{-x_2-x_3}\partial_1 \\ e^{-x_2-x_3}(1 - e^{x_3})\partial_1 + \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ -x_1\partial_1 + \partial_2 \\ -(1 - e^{-x_2} + x_1)\partial_1 + \partial_3 \\ \partial_4 \end{pmatrix}$
$VI_0 \oplus R$	$\begin{pmatrix} \cosh(x_3)\partial_1 - \sinh(x_3)\partial_2 \\ \cosh(x_3)\partial_2 - \sinh(x_3)\partial_1 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 - x_1\partial_2 + \partial_3 \\ \partial_4 \end{pmatrix}$
$(VI_0 \oplus R).ix$	$\begin{pmatrix} \cosh(x_4)\partial_1 - \sinh(x_4)\partial_2 \\ \cosh(x_4)\partial_2 - \sinh(x_4)\partial_1 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -x_2\partial_1 - x_1\partial_2 + \partial_4 \end{pmatrix}$
$(VI_0 \oplus R).iv$	$\begin{pmatrix} \partial_1 + x_4\partial_2 + x_2\partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \cosh(x_1)\partial_2 + \sinh(x_1)\partial_4 \\ \partial_3 \\ \cosh(x_1)\partial_4 + \sinh(x_1)\partial_4 \end{pmatrix}$
$(VI_0 \oplus R).v$	$\begin{pmatrix} \partial_1 - x_2\partial_2 + x_4\partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{-x_1}\partial_2 \\ \partial_3 \\ e^{x_1}\partial_4 \end{pmatrix}$

**Table 5 :** (Continued.)

$VII_0 \oplus R$	$\begin{pmatrix} \cos(x_3)\partial_1 + \sin(x_3)\partial_2 \\ \cos(x_3)\partial_2 - \sin(x_3)\partial_1 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 + x_1\partial_2 + \partial_3 \\ \partial_4 \end{pmatrix}$
$(VII_0 \oplus R).iv$	$\begin{pmatrix} \cos(x_4)\partial_1 - \sin(x_4)\partial_2 \\ \cos(x_4)\partial_2 + \sin(x_4)\partial_1 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ x_2\partial_1 - x_1\partial_2 + \partial_4 \end{pmatrix}$
$(VII_0 \oplus R).ii$	$\begin{pmatrix} \partial_1 - x_4\partial_2 + x_2\partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \cos(x_1)\partial_2 + \sin(x_1)\partial_4 \\ \partial_3 \\ \cos(x_1)\partial_4 - \sin(x_1)\partial_2 \end{pmatrix}$
$A_{4,11}^b$	$\begin{pmatrix} e^{-2bx_4}\partial_1 \\ e^{-bx_4}((-x_3\partial_1 + \partial_2)\cos(x_4) + \sin(x_4)\partial_3) \\ e^{-bx_4}((x_3\partial_1 - \partial_2)\sin(x_4) + \cos(x_4)\partial_3) \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_2\partial_1 + \partial_3 \\ \frac{x_3^3 - x_2^2 - 4bx_1}{2}\partial_1 - (x_3 + bx_2)\partial_2 + (x_2 - bx_3)\partial_3 + \partial_4 \end{pmatrix}$
$A_{4,11}^b.i$	$\begin{pmatrix} \partial_1 - \frac{bx_2+x_3}{2b}\partial_2 + \frac{x_2-x_3b}{2b}\partial_3 - \frac{x_2^2-x_3^2+2x_4}{2}\partial_4 \\ \partial_2 - 2bx_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{-\frac{x_1}{2}(\cos(\frac{x_1}{2b})\partial_2 + (\partial_3 - 2be^{x_1}x_2\partial_4)\sin(\frac{x_1}{2b}))} \\ -e^{-\frac{x_1}{2}(\sin(\frac{x_1}{2b})\partial_2 - (\partial_3 - 2be^{x_1}x_2\partial_4)\cos(\frac{x_1}{2b}))} \\ e^{-x_1}\partial_4 \end{pmatrix}$
$A_{4,12}$	$\begin{pmatrix} e^{-x_3}(\cos(x_4)\partial_1 + \sin(x_4)\partial_2) \\ e^{-x_3}(\cos(x_4)\partial_2 - \sin(x_4)\partial_1) \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_1\partial_1 - x_2\partial_2 + \partial_3 \\ -x_2\partial_1 + x_1\partial_2 + \partial_4 \end{pmatrix}$
$A_{4,12}.ii$	$\begin{pmatrix} \partial_1 + (1 - e^{x_2} - x_4)\partial_3 + x_3\partial_4 \\ \partial_2 + x_3\partial_3 + x_4\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 - qe^{x_2}((1 - \cos x_1)\partial_4 + \sin x_1\partial_3) \\ e^{x_2}(\cos x_1\partial_3 + \sin x_1\partial_4) \\ e^{x_2}(\cos x_1\partial_4 - \sin x_1\partial_3) \end{pmatrix}$
$A_{4,6}^{a,0}$	$\begin{pmatrix} e^{-ax_4}\partial_1 \\ \cos(x_4)\partial_2 + \sin(x_4)\partial_3 \\ \cos(x_4)\partial_3 - \sin(x_4)\partial_2 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \\ -ax_1\partial_1 - x_3\partial_2 + x_2\partial_3 + \partial_4 \end{pmatrix}$
$A_{4,6}^{a,0}.i$	$\begin{pmatrix} \partial_1 + x_3\partial_2 - x_2\partial_3 + ax_4\partial_4 \\ \partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \cos(x_1)\partial_2 - \sin(x_1)\partial_3 \\ \cos(x_1)\partial_3 + \sin(x_1)\partial_2 \\ e^{ax_1}\partial_4 \end{pmatrix}$
$A_{4,9}^b$	$\begin{pmatrix} e^{-(1+b)x_4}\partial_1 \\ e^{-x_4}(\partial_2 - x_3\partial_1) \\ e^{-bx_4}\partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 - x_2\partial_1 \\ -(1+b)x_1\partial_1 - x_2\partial_2 - bx_3\partial_3 + \partial_4 \end{pmatrix}$
$A_{4,9}^b.i$	$\begin{pmatrix} \partial_1 - bx_2\partial_2 - x_3\partial_3 - (1+b)x_4\partial_4 \\ \partial_2 - (1+b)^2x_3\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{-bx_1}\partial_2 \\ e^{-x_1}(\partial_3 - (1+b)^2x_2\partial_4) \\ e^{-(1+b)x_1}\partial_4 \end{pmatrix}$
$A_{4,9}^1.ii$	$\begin{pmatrix} e^{x_3}\partial_1 - 2e^{2x_3}x_4\partial_2 \\ e^{2x_3}\partial_2 \\ \partial_3 - x_4\partial_4 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ x_1\partial_1 + 2x_2\partial_2 + \partial_3 \\ -2x_1\partial_2 + e^{-x_3}\partial_4 \end{pmatrix}$
$A_2 \oplus A_2$	$\begin{pmatrix} \partial_1 - x_2\partial_2 \\ \partial_2 \\ \partial_3 - x_4\partial_4 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ e^{-x_1}\partial_2 \\ \partial_3 \\ e^{-x_3}\partial_4 \end{pmatrix}$
$(A_2 \oplus A_2).v$	$\begin{pmatrix} \partial_1 - qx_4\partial_2 \\ e^{-qx_3}\partial_2 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -qx_2\partial_2 + \partial_3 \\ e^{-qx_1}\partial_4 \end{pmatrix}$
$(A_2 \oplus A_2).i$	$\begin{pmatrix} e^{-x_3}\partial_1 \\ \partial_2 - x_4\partial_4 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 \\ -x_1\partial_1 + \partial_3 \\ e^{-x_2}\partial_4 \end{pmatrix}$
$(A_2 \oplus A_2).ii$	$\begin{pmatrix} \partial_1 + (1 - e^{-x_2})\partial_3 - x_4\partial_4 \\ \partial_2 - x_3\partial_3 \\ \partial_3 \\ \partial_4 \end{pmatrix}$	$\begin{pmatrix} \partial_1 \\ \partial_2 + e^{-x_2}x_1\partial_3 \\ e^{-x_2}\partial_3 \\ e^{-x_1}\partial_4 \end{pmatrix}$

## Appendix D:

**Table 6:** Poisson brackets over Poisson-Lie group  $G$  (using Sklyanin bracket) .

$\mathbf{g}$	$\mathbf{\bar{g}}$	Non-zero Poisson brackets relations
$A_{4,1}$	$4A_1$	
	$A_{4,1}.iii$	$\{x_1, x_2\} = x_3 - \frac{x_4^4}{2}, \quad \{x_1, x_3\} = x_4$
	$(II \oplus R).i$	$\{x_1, x_2\} = x_3 + x_4$
	$(II \oplus R).ii$	$\{x_1, x_2\} = x_4$
	$(II \oplus R).iv$	$\{x_1, x_2\} = -\frac{x_4^2}{2}, \quad \{x_1, x_3\} = x_4$
$A_{4,2}^{-1}$	$4A_1$	
	$A_{4,2}^{-1}.i$	$\{x_1, x_2\} = -x_1 - x_4, \quad \{x_2, x_3\} = -x_3, \quad \{x_2, x_4\} = 1 - e^{-x_4}$
	$A_{4,5}^{-1,-1}.i$	$\{x_1, x_2\} = -x_1, \quad \{x_2, x_3\} = -x_3, \quad \{x_2, x_4\} = 1 - e^{-x_4}$
	$(II \oplus R).iii$	$\{x_2, x_3\} = \frac{1 - e^{-2x_4}}{2}$
	$(II \oplus R).v$	$\{x_1, x_2\} = x_4, \quad \{x_2, x_3\} = \frac{1 - e^{-2x_4}}{2}$
	$(II \oplus R).ii$	$\{x_1, x_2\} = x_4$
$A_{4,3}$	$4A_1$	
	$A_{4,3}.ii$	$\{x_1, x_2\} = -x_3, \quad \{x_1, x_4\} = 1 - e^{-x_4}$
	$(II \oplus R).iv$	$\{x_1, x_2\} = -1 + e^{-x_4}(1 + x_4), \quad \{x_1, x_3\} = 1 - e^{-x_4}$
	$(III \oplus R).i$	$\{x_1, x_2\} = x_1$
$A_{4,5}^{-1,-1}$	$4A_1$	
	$A_{4,5}^{-1,-1}.i$	$\{x_1, x_2\} = -x_1, \quad \{x_2, x_3\} = -x_3, \quad \{x_2, x_4\} = -1 + e^{x_4}$
	$(II \oplus R).vii$	$\{x_2, x_3\} = \frac{1 - e^{2x_4}}{2}$
$A_{4,5}^{-1,b}$	$4A_1$	
	$A_{4,5}^{-1,b}.i$	$\{x_1, x_3\} = x_1, \quad \{x_2, x_3\} = -x_2, \quad \{x_3, x_4\} = 1 - e^{-bx_4}$
	$(II \oplus R).iv$	$\{x_1, x_3\} = \frac{1 - e^{-(1+b)x_4}}{1+b}$
	$(II \oplus R).iii$	$\{x_2, x_3\} = \frac{-1 + e^{(1-b)x_4}}{1-b}$
$A_{4,5}^{a,-1}$	$4A_1$	
	$A_{4,5}^{a,-1}.i$	$\{x_1, x_2\} = x_1, \quad \{x_2, x_3\} = x_3, \quad \{x_2, x_4\} = 1 - e^{-ax_4}$
	$(II \oplus R).iii$	$\{x_2, x_3\} = \frac{-1 + e^{(1-a)x_4}}{1-a}$
	$(II \oplus R).v$	$\{x_1, x_2\} = \frac{1 - e^{-(1+a)x_4}}{1+a}, \quad \{x_2, x_3\} = \frac{-1 + e^{(1-a)x_4}}{1-a}$
$A_{4,5}^{a,-a}$	$4A_1$	
	$A_{4,5}^{a,-a}.i$	$\{x_1, x_2\} = -ax_2, \quad \{x_1, x_3\} = ax_3, \quad \{x_1, x_4\} = 1 - e^{-x_4}$
	$(II \oplus R).iv$	$\{x_1, x_3\} = \frac{1 - e^{(a-1)x_4}}{1-a}$
	$(II \oplus R).viii$	$\{x_1, x_2\} = \frac{1 - e^{-(a+1)x_4}}{1+a}, \quad \{x_1, x_3\} = \frac{1 - e^{(a-1)x_4}}{1-a}$
$A_{4,6}^{a,0}$	$4A_1$	
	$A_{4,6}^{a,0}.i$	$\{x_1, x_2\} = x_3, \quad \{x_1, x_3\} = -x_2, \quad \{x_1, x_4\} = -1 + e^{-ax_4}$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{a}{1+a^2} - \frac{e^{-ax_4}}{1+a^2}(a \cos(x_4) - \sin(x_4)), \quad \{x_1, x_3\} = \frac{1}{1+a^2} - \frac{e^{-ax_4}}{1+a^2}(\cos(x_4) + a \sin(x_4))$
$A_{4,7}$	$4A_1$	
	$A_{4,7}.i$	$\{x_1, x_2\} = \frac{x_2 - x_3}{2}, \quad \{x_1, x_3\} = -\frac{x_3}{2} + e^{-2x_4}x_3, \quad \{x_1, x_4\} = \frac{1 - e^{-2x_4}}{2}, \quad \{x_2, x_3\} = 1 - e^{-2x_4}$
	$A_{4,9}^{-\frac{1}{2}}.i$	$\{x_1, x_2\} = 2x_1 - \frac{1}{2}x_3^2, \quad \{x_1, x_4\} = e^{-x_4}x_3, \quad \{x_2, x_3\} = -x_3, \quad \{x_2, x_4\} = 1 - e^{-x_4}$
	$(II \oplus R).iv$	$\{x_1, x_2\} = \frac{1}{9}(-1 + e^{-3x_4}(1 + 3x_4)), \quad \{x_1, x_3\} = \frac{1}{3}(1 - e^{-3x_4})$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{1}{3}(1 - e^{-3x_4})$
$A_{4,9}^0$	$4A_1$	
	$A_{4,9}^0.iv$	$\{x_1, x_2\} = \frac{1}{2}(1 - e^{-2x_4}), \quad \{x_1, x_3\} = e^{-x_4}x_3, \quad \{x_1, x_4\} = 1 - e^{-x_4}, \quad \{x_2, x_3\} = 1 - e^{-x_4}$
	$(II \oplus R).viii$	$\{x_1, x_2\} = \frac{1 - e^{-2x_4}}{2}, \quad \{x_1, x_3\} = 1 - e^{-x_4}$
	$(II \oplus R).iv$	$\{x_1, x_2\} = 1 - e^{-x_4}$
	$(VI_0 \oplus R).v$	$\{x_1, x_2\} = x_2, \quad \{x_1, x_4\} = -1 + e^{-x_4}$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{1}{2}(1 - e^{-2x_4})$

**Table 6:** (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{4,9}^{-\frac{1}{2}}$	$4A_1$	
	$A_{4,9}^{-\frac{1}{2}}.iii$	$\{x_1, x_2\} = -2x_2, \{x_1, x_3\} = 2x_3 + 2e^{-\frac{x_4}{2}}x_3, \{x_1, x_4\} = 4 - 4e^{-\frac{x_4}{2}}, \{x_2, x_3\} = 2 - 2e^{-\frac{x_4}{2}}$
	$A_{4,9}^{-\frac{1}{2}}.iv$	$\{x_1, x_2\} = \frac{x_1}{2}, \{x_1, x_4\} = x_3e^{-x_4}, \{x_2, x_3\} = \frac{x_3}{2}, \{x_2, x_4\} = 1 - e^{-x_4}$
	$A_{4,9}^1.ii$	$\{x_1, x_2\} = -2x_2^2, \{x_1, x_3\} = x_2x_3 - x_1, \{x_1, x_4\} = 2x_2, \{x_2, x_3\} = -2x_2, \{x_3, x_4\} = 2e^{\frac{x_4}{2}} - 2$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{2}{3}(1 - e^{-\frac{3x_4}{2}})$
$A_{4,9}^1$	$4A_1$	
	$A_{4,9}^1.i$	$\{x_1, x_2\} = \frac{-x_2}{4}, \{x_1, x_3\} = \frac{x_3}{4}(1 - 2e^{-2x_4}), \{x_4, x_1\} = \frac{1}{4}(1 - e^{-2x_4}), \{x_3, x_2\} = \frac{1}{2}(1 - e^{-2x_4})$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{1}{3}(1 - e^{-3x_4})$
$A_{4,9}^b$	$4A_1$	
	$A_{4,9}^b.i$	$\{x_1, x_2\} = bx_2, \{x_1, x_3\} = x_3(-b + e^{-(1+b)x_4}(1+b)), \{x_1, x_4\} = 1 - e^{-(1+b)x_4},$ $\{x_2, x_3\} = (1+b)(1 - e^{-(1+b)x_4})$
	$(II \oplus R).iv$	$\{x_1, x_3\} = \frac{1}{1+2b}(1 - e^{-(1+2b)x_4})$
$A_{4,11}^b$	$4A_1$	
	$A_{4,11}^b.i$	$\{x_1, x_2\} = \frac{1}{2} - \frac{x_3}{2b}, \{x_1, x_3\} = \frac{x_2}{2b} - \frac{x_3}{2} + x_3e^{-2bx_4}, \{x_1, x_4\} = \frac{1}{2b} - \frac{e^{-2bx_4}}{2b},$ $\{x_2, x_3\} = 1 - e^{-2bx_4}$
	$(II \oplus R).ii$	$\{x_1, x_2\} = \frac{1}{1+9b^2}(3b - e^{-3bx_4}(3b\cos(x_4) - \sin(x_4))),$ $\{x_1, x_3\} = \frac{1}{1+9b^2}(1 - e^{-3bx_4}(\cos(x_4) + 3b\sin(x_4)))$
$A_{4,12}$	$4A_1$	
	$A_{4,12}.ii$	$\{x_1, x_2\} = \frac{q}{2} - \frac{1}{2}qe^{-2x_3}, \{x_1, x_3\} = -e^{-x_3}\sin(x_4), \{x_1, x_4\} = 1 - e^{-x_3}\cos(x_4)$ $\{x_2, x_3\} = -1 + e^{-x_3}\cos(x_4), \{x_2, x_4\} = -e^{-x_3}\sin(x_4)$
	$(II \oplus R).xii$	$\{x_1, x_2\} = \frac{1}{2}(1 - e^{-2x_3})$
$(A_2 \oplus A_2)$	$4A_1$	
	$(A_2 \oplus A_2).v$	$\{x_1, x_4\} = qx_4, \{x_2, x_3\} = qx_2, \{x_2, x_4\} = -qx_2x_4$
	$(II \oplus R).xiii$	$\{x_2, x_4\} = 1 - e^{-x_1-x_3}$
	$(III \oplus R).i$	$\{x_1, x_2\} = 1 - e^{-x_1}$
	$(III \oplus R).iii$	$\{x_3, x_4\} = 1 - e^{-x_3}$
	$(VI_0 \oplus R).vii$	$\{x_1, x_4\} = -1 + e^{-x_3}, \{x_2, x_4\} = 1 - e^{-x_1-x_3} + x_2, \{x_3, x_4\} = -1 + e^{-x_3}$
	$(VI_0 \oplus R).viii$	$\{x_1, x_4\} = -1 + e^{-x_3}, \{x_2, x_4\} = x_2, \{x_3, x_4\} = -1 + e^{-x_3}$
$VII_0 \oplus R$	$4A_1$	
	$(VII_0 \oplus R).iv$	$\{x_1, x_4\} = x_2, \{x_2, x_4\} = -x_1$
	$(II \oplus R).xiv$	$\{x_1, x_4\} = \sin(x_3), \{x_2, x_4\} = 1 - \cos(x_3)$
$VI_0 \oplus R$	$4A_1$	
	$(VI_0 \oplus R).ix$	$\{x_1, x_4\} = x_2, \{x_2, x_4\} = x_1$
	$(II \oplus R).xiv$	$\{x_1, x_4\} = \sinh(x_3), \{x_2, x_4\} = 1 - \cosh(x_3)$
$III \oplus R$	$4R$	
	$(III \oplus R).xiii$	$\{x_1, x_2\} = \frac{1}{2}(-1 + e^{2x_1}), \{x_1, x_3\} = \frac{1}{2}(-1 + e^{2x_1}), \{x_2, x_3\} = \frac{1}{2}(-1 + e^{2x_1})$
$II \oplus R$	$4A_1$	
	$(II \oplus R).xiv$	$\{x_1, x_4\} = x_3$
$A_{4,1}.iii$	$A_{4,1}$	$\{x_2, x_4\} = x_1, \{x_3, x_4\} = -\frac{x_2^2}{2} + x_2$
$A_{4,2}^{-1}.i$	$A_{4,2}^{-1}$	$\{x_1, x_4\} = -e^{-x_2}x_1, \{x_2, x_4\} = 1 - e^{-x_2}, \{x_3, x_4\} = x_2 + x_3$
$A_{4,3}.ii$	$A_{4,3}$	$\{x_1, x_4\} = 1 - e^{-x_1}, \{x_3, x_4\} = x_2$
$A_{4,5}^{-1,-1}.i$	$A_{4,5}^{-1,-1}$	$\{x_1, x_4\} = e^{-x_2}x_1, \{x_2, x_4\} = -1 + e^{-x_2}, \{x_3, x_4\} = -x_3$
$A_{4,5}^{-1,b}.i$	$A_{4,5}^{-1,b}$	$\{x_1, x_4\} = e^{-bx_3}x_1, \{x_2, x_4\} = -e^{-bx_3}x_2, \{x_3, x_4\} = 1 - e^{-bx_3}$
$A_{4,5}^{a,-1}.i$	$A_{4,5}^{a,-1}$	$\{x_1, x_4\} = e^{-ax_2}x_1, \{x_2, x_4\} = 1 - e^{-ax_2}, \{x_3, x_4\} = -x_3$
$A_{4,5}^{a,-a}.i$	$A_{4,5}^{a,-a}$	$\{x_1, x_4\} = 1 - e^{-x_1}, \{x_2, x_4\} = -ax_2, \{x_3, x_4\} = -ax_3$
$A_{4,6}^{a,0}.i$	$A_{4,6}^{a,0}$	$\{x_1, x_4\} = -1 + e^{ax_1}, \{x_2, x_4\} = -x_3, \{x_3, x_4\} = x_2$
$A_{4,7}.i$	$A_{4,7}$	$\{x_1, x_4\} = 2 - 2e^{-x_1}, \{x_2, x_3\} = 1 - e^{-x_1}, \{x_2, x_4\} = -x_2(1 - 2e^{-x_1}), \{x_3, x_4\} = x_2 + x_3$

**Table 6:** (Continued.)

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{4,9}^0.iv$	$A_{4,9}^0$	$\{x_1, x_4\} = 1 - e^{-x_1}$ , $\{x_2, x_3\} = 1 - e^{-x_1}$ , $\{x_2, x_4\} = e^{-x_1}x_2$ , $\{x_3, x_4\} = \frac{1}{2} + \frac{e^{-2x_1}}{2} - e^{-x_1}$
$(VI_0 \oplus R).v$	$A_{4,9}^0$	$\{x_1, x_4\} = -1 + e^{x_1}$ , $\{x_2, x_3\} = 1 - e^{-x_1}$ , $\{x_2, x_4\} = x_2$
$A_{4,9}^{-\frac{1}{2}.iii}$	$A_{4,9}^{-\frac{1}{2}}$	$\{x_1, x_4\} = \frac{1}{4}(1 - e^{-2x_1})$ , $\{x_2, x_3\} = \frac{1}{2}(1 - e^{-2x_1})$ , $\{x_2, x_4\} = \frac{x_2}{2}(1 + e^{-2x_1})$ , $\{x_3, x_4\} = -\frac{x_3}{2}$
$A_{4,9}^{-\frac{1}{2}.iv}$	$A_{4,9}^{-\frac{1}{2}}$	$\{x_1, x_3\} = -\frac{1}{2}e^{-\frac{x_2}{2}}x_1^2$ , $\{x_1, x_4\} = \frac{1}{2}e^{-x_2}x_1$ , $\{x_2, x_3\} = e^{-\frac{x_2}{2}}x_1$ , $\{x_2, x_4\} = 1 - e^{x_2}$ , $\{x_3, x_4\} = -\frac{x_3}{2}$
$A_{4,9}^1.ii$	$A_{4,9}^{-\frac{1}{2}}$	$\{x_1, x_2\} = -x_1^2$ , $\{x_1, x_4\} = \frac{1}{2}e^{-x_3}x_1$ , $\{x_2, x_3\} = x_1$ , $\{x_2, x_4\} = e^{-x_3}x_2$ , $\{x_3, x_4\} = \frac{-1}{2}(1 - e^{-x_3})$
$A_{4,9}^1.i$	$A_{4,9}^1$	$\{x_1, x_4\} = -4(1 - e^{\frac{x_1}{2}})$ , $\{x_2, x_3\} = -2(1 - e^{\frac{x_1}{2}})$ , $\{x_2, x_4\} = -x_2(1 - 2e^{\frac{x_1}{2}})$ , $\{x_3, x_4\} = x_3$
$A_{4,9}^b.i$	$A_{4,9}^b$	$\{x_1, x_4\} = 1 - e^{-(1+b)x_1}$ , $\{x_2, x_3\} = \frac{1}{1+b}(1 - e^{-(1+b)x_1})$ , $\{x_2, x_4\} = -x_2(b - (1+b)e^{-(1+b)x_1})$ , $\{x_3, x_4\} = bx_3$
$A_{4,11}^b.i$	$A_{4,11}^b$	$\{x_1, x_4\} = 2a(1 - e^{-x_1})$ , $\{x_2, x_3\} = 1 - e^{-x_1}$ , $\{x_2, x_4\} = ax_2 - x_3$ , $\{x_3, x_4\} = x_2 + ax_3$
$A_{4,12}.i$	$A_{4,12}$	$\{x_1, x_3\} = e^{x_2}\sin(x_1)$ , $\{x_1, x_4\} = 1 - e^{x_2}\cos(x_1)$ , $\{x_2, x_3\} = -1 + e^{x_2}\cos(x_1)$ , $\{x_2, x_4\} = e^{x_2}\sin(x_1)$ , $\{x_3, x_4\} = \frac{q}{2}(1 - 2e^{x_2} - e^{2x_2} + 2e^{2x_2}\cos(x_1))$
$(A_2 \oplus A_2).v$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2$ , $\{x_3, x_4\} = x_4$
$(VII_0 \oplus R).iv$	$VII_0 \oplus R$	$\{x_1, x_3\} = -x_2$ , $\{x_2, x_3\} = x_1$
$(VI_0 \oplus R).ix$	$VI_0 \oplus R$	$\{x_1, x_3\} = x_2$ , $\{x_2, x_3\} = x_1$
$(III \oplus R).xiii$	$III \oplus R$	$\{x_1, x_2\} = -1 + e^{-x_2-x_3}$ , $\{x_1, x_3\} = -1 + e^{-x_2-x_3}$
$(II \oplus R).xiv$	$II \oplus R$	$\{x_2, x_3\} = x_1$
$(A_2 \oplus A_2).i$	$A_{4,3}$	$\{x_1, x_4\} = e^{-x_2}x_1$ , $\{x_3, x_4\} = 1 - e^{-x_2}$
$(A_2 \oplus A_2).ii$	$A_{4,3}$	$\{x_1, x_4\} = 1 - e^{-x_1}$ , $\{x_3, x_4\} = 1 - e^{-x_2}$
$(VI_0 \oplus R).iv$	$A_{4,9}^0$	$\{x_1, x_2\} = -1 + \cosh(x_1)$ , $\{x_1, x_4\} = \sinh(x_1)$ , $\{x_2, x_3\} = \sinh(x_1)$ , $\{x_2, x_4\} = x_2$ , $\{x_3, x_4\} = 1 - \cosh(x_1)$
$(VII_0 \oplus R).ii$	$A_{4,9}^0$	$\{x_1, x_2\} = -1 + \cos(x_1)$ , $\{x_1, x_4\} = \sin(x_1)$ , $\{x_2, x_3\} = \sin(x_1)$ , $\{x_2, x_4\} = x_2$ , $\{x_3, x_4\} = 1 - \cos(x_1)$

**Table 7:** Poisson brackets over Poisson-Lie group  $G$  (using  $\pi(g)$ ).

$\mathbf{g}$	$\tilde{\mathbf{g}}$	Non-zero Poisson brackets relations
$A_{4,1}$	$A_{4,1}.i$	$\{x_1, x_2\} = x_3 + \frac{x_4^3}{6}$ , $\{x_1, x_3\} = -\frac{x_4^2}{2}$ , $\{x_2, x_3\} = x_4$
	$A_{4,1}.ii$	$\{x_1, x_2\} = -\frac{x_3^2}{2} + x_4$ , $\{x_1, x_4\} = x_3$
	$A_{4,9}^0.i$	$\{x_1, x_2\} = x_1 + x_2 + \frac{qx_4^3}{6}$ , $\{x_1, x_3\} = -\frac{qx_4^2}{2}$ , $\{x_1, x_4\} = x_4 + \frac{x_4^2}{2}$ , $\{x_2, x_3\} = qx_4$ , $\{x_2, x_4\} = -x_4$
	$A_{4,9}^0.ii$	$\{x_1, x_2\} = x_1 - \frac{x_4^2}{2}$ , $\{x_1, x_3\} = x_4$ , $\{x_1, x_4\} = \frac{x_4^2}{2}$ , $\{x_2, x_4\} = -x_4$
	$A_{4,9}^0.iii$	$\{x_1, x_2\} = x_2 + \frac{x_4^3}{6}$ , $\{x_1, x_3\} = -\frac{x_4^2}{2}$ , $\{x_1, x_4\} = x_4$ , $\{x_2, x_3\} = x_4$
	$(II \oplus R).iii$	$\{x_1, x_2\} = \frac{x_4^3}{6}$ , $\{x_1, x_3\} = -\frac{x_4^2}{2}$ , $\{x_2, x_3\} = x_4$
	$(VI_0 \oplus R).i$	$\{x_1, x_2\} = -\frac{x_3^2 + x_4^2}{2}$ , $\{x_1, x_3\} = x_4$ , $\{x_1, x_4\} = x_3$
	$(VI_0 \oplus R).ii$	$\{x_1, x_3\} = x_3$ , $\{x_1, x_4\} = -x_4$
	$(VII_0 \oplus R).i$	$\{x_1, x_2\} = \frac{x_4^2 - x_3^2}{2}$ , $\{x_1, x_3\} = -x_4$ , $\{x_1, x_4\} = x_3$
$A_{4,2}^{-1}$	$(II \oplus R).vi$	$\{x_1, x_2\} = -\frac{x_4^2}{2}$ , $\{x_1, x_3\} = x_4$ , $\{x_2, x_3\} = \frac{1 - e^{-2x_4}}{2}$
	$(II \oplus R).iv$	$\{x_1, x_2\} = -\frac{x_4^2}{2}$ , $\{x_1, x_3\} = x_4$
$A_{4,3}$	$(A_2 \oplus A_2).i$	$\{x_1, x_3\} = x_1$ , $\{x_2, x_4\} = x_4$
	$(A_2 \oplus A_2).ii$	$\{x_1, x_2\} = -x_3$ , $\{x_1, x_4\} = 1 - e^{-x_4}$ , $\{x_2, x_3\} = x_3$
	$(A_2 \oplus A_2).iii$	$\{x_1, x_2\} = -qx_1x_3$ , $\{x_1, x_4\} = qx_1$ , $\{x_2, x_3\} = x_3$ , $\{x_2, x_4\} = -qx_3$
	$A_{4,3}.i$	$\{x_1, x_2\} = 1 - e^{-x_4}$ , $\{x_2, x_3\} = x_3$
	$A_{4,3}.iii$	$\{x_1, x_2\} = -qx_1x_3$ , $\{x_1, x_4\} = qx_1$ , $\{x_2, x_4\} = -qx_3$
	$A_{4,3}.iv$	$\{x_1, x_2\} = x_1$ , $\{x_2, x_3\} = qx_4$
	$(II \oplus R).iii$	$\{x_2, x_3\} = x_4$

**Table 7:** (Continued.)

<b>g</b>	<b><math>\tilde{g}</math></b>	<b>Non-zero Poisson brackets relations</b>
$A_{4,5}^{-1,-1}$	$(II \oplus R).v$	$\{x_1, x_2\} = 1 - e^{-x_4}, \quad \{x_2, x_3\} = x_4$
	$(II \oplus R).vi$	$\{x_1, x_2\} = -1 + e^{-x_4}(1 + x_4), \quad \{x_1, x_3\} = 1 - e^{-x_4}, \quad \{x_2, x_3\} = x_4$
	$(III \oplus R).ii$	$\{x_2, x_3\} = x_3$
	$(VI_0 \oplus R).iii$	$\{x_1, x_2\} = x_1, \quad \{x_2, x_3\} = x_3$
	$(II \oplus R).ii$	$\{x_1, x_2\} = x_4$
	$(II \oplus R).viii$	$\{x_1, x_2\} = x_4, \quad \{x_1, x_3\} = \frac{1 - e^{-(1+b)x_4}}{1+b}$
$A_{4,5}^{-1,b}$	$(II \oplus R).v$	$\{x_1, x_2\} = x_4, \quad \{x_2, x_3\} = \frac{-1 + e^{(1-b)x_4}}{1-b}$
	$(II \oplus R).iv$	$\{x_1, x_3\} = x_4$
	$(II \oplus R).ix$	$\{x_1, x_2\} = \frac{1 - e^{-(1+a)x_4}}{1+a}, \quad \{x_1, x_3\} = x_4, \quad \{x_2, x_3\} = \frac{-1 + e^{(1-a)x_4}}{1-a}$
$A_{4,5}^{a,-1}$	$(II \oplus R).viii$	$\{x_1, x_2\} = \frac{1 - e^{-(1+a)x_4}}{1+a}, \quad \{x_1, x_3\} = x_4,$
	$(II \oplus R).iii$	$\{x_2, x_3\} = x_4$
	$(II \oplus R).v$	$\{x_1, x_2\} = \frac{1 - e^{-(1+a)x_4}}{1+a}, \quad \{x_2, x_3\} = x_4$
$A_{4,6}^{a,0}$	$(II \oplus R).iii$	$\{x_2, x_3\} = x_4$
$A_{4,9}^0$	$A_{4,9}^0.v$	$\{x_1, x_2\} = -\frac{(q_1 - q_2)}{2}x_2^2, \quad \{x_1, x_3\} = q_1x_1 + x_2, \quad \{x_1, x_4\} = q_2x_2, \quad \{x_2, x_3\} = q_1x_2$
	$(II \oplus R).x$	$\{x_1, x_3\} = 1 - e^{-x_4} + x_2$
	$(II \oplus R).xi$	$\{x_1, x_2\} = \frac{1}{2}(-1 + e^{-2x_4} - 2x_2 - x_2^2), \quad \{x_1, x_4\} = 1 - e^{-x_4} + x_2$
	$(VI_0 \oplus R).iv$	$\{x_1, x_2\} = -\frac{1}{2}(1 - e^{-2x_4} - x_2^2), \quad \{x_1, x_4\} = -x_2$
	$(VII_0 \oplus R).ii$	$\{x_1, x_2\} = -\frac{1}{2}(1 - e^{-2x_4} + x_2^2), \quad \{x_1, x_4\} = x_2$
	$A_{4,9}^{-\frac{1}{2}}.ii$	$\{x_1, x_2\} = -2x_2, \quad \{x_1, x_3\} = 2x_3 + 2e^{-\frac{x_4}{2}}x_3 + x_4, \quad \{x_1, x_4\} = 4 - 4e^{-\frac{x_4}{2}}, \quad \{x_2, x_3\} = 2 - 2e^{-\frac{x_4}{2}}$
$A_{4,12}$	$(II \oplus R).iv$	$\{x_1, x_3\} = x_4$
	$A_{4,12}.i$	$\{x_1, x_2\} = \frac{q_2}{2}(x_1^2 + x_2^2), \quad \{x_1, x_3\} = -q_1x_1, \quad \{x_1, x_4\} = q_2x_1 + q_1x_2,$ $\{x_2, x_3\} = -q_1x_2, \quad \{x_2, x_4\} = -q_1x_1 + q_2x_2$
	$A_{4,12}.iii$	$\{x_1, x_3\} = -qx_1, \quad \{x_1, x_4\} = qx_2, \quad \{x_2, x_3\} = -qx_2, \quad \{x_2, x_4\} = -qx_1$
	$(VII_0 \oplus R).iii$	$\{x_1, x_2\} = \frac{q}{2}(-1 + e^{-2x_3} - 2x_1 - x_1^2 - x_2^2), \quad \{x_1, x_3\} = q(x_2 - e^{-x_3}\sin(x_4))$ $\{x_2, x_3\} = -q(1 + x_1 - e^{-x_3}\cos(x_4))$
	$(A_2 \oplus A_2).iv$	$\{x_1, x_2\} = q_1x_2, \quad \{x_1, x_4\} = q_2x_4, \quad \{x_2, x_3\} = q_2x_2, \quad \{x_2, x_4\} = -q_2x_2x_4$
	$(A_2 \oplus A_2).vi$	$\{x_1, x_2\} = qx_2, \quad \{x_3, x_4\} = 1 + e^{-x_3}$
$(A_2 \oplus A_2).vii$	$(A_2 \oplus A_2).vii$	$\{x_1, x_4\} = q_1x_4, \quad \{x_2, x_3\} = q_1x_2, \quad \{x_2, x_4\} = -q_1x_2x_4, \quad \{x_3, x_4\} = q_2x_4$
	$(III \oplus R).iv$	$\{x_1, x_2\} = 1 - e^{-x_1} + qx_2$
	$(III \oplus R).v$	$\{x_3, x_4\} = 1 - e^{-x_3} + qx_4$
	$(VI_0 \oplus R).vi$	$\{x_1, x_2\} = -qx_2, \quad \{x_1, x_4\} = qx_4, \quad \{x_2, x_3\} = qx_2, \quad \{x_2, x_4\} = -qx_2x_4, \quad \{x_3, x_4\} = qx_4$
	$(III \oplus R).vi$	$\{x_1, x_4\} = 1 - e^{-x_3} + x_1 + x_2, \quad \{x_2, x_4\} = 1 - e^{-x_3} + x_1 + x_2$
	$(III \oplus R).vii$	$\{x_1, x_4\} = -1 + e^{x_3} + x_1 - x_2, \quad \{x_2, x_4\} = 1 - e^{x_3} - x_1 + x_2$
$VI_0 \oplus R$	$(II \oplus R).xii$	$\{x_1, x_2\} = x_3$
	$(II \oplus R).i$	$\{x_1, x_2\} = x_3 + x_4$
	$(VII_0 \oplus R).4A_1$	
	$(II \oplus R).xii$	$\{x_1, x_2\} = x_3$
	$(II \oplus R).i$	$\{x_1, x_2\} = x_3 + x_4$
	$(III \oplus R).vix$	$\{x_2, x_4\} = x_4, \quad \{x_3, x_4\} = -x_4$
$III \oplus R$	$(II \oplus R).xv$	$\{x_2, x_3\} = -\frac{1}{2}(1 - e^{2x_1}), \quad \{x_2, x_4\} = -\frac{1}{4}(1 - e^{2x_1} + 2x_1), \quad \{x_3, x_4\} = -\frac{1}{4}(1 - e^{2x_1} - 2x_1)$
	$(II \oplus R).xvi$	$\{x_1, x_2\} = x_3, \quad \{x_1, x_4\} = x_3$
	$(II \oplus R).xvi$	$\{x_1, x_2\} = x_3, \quad \{x_1, x_4\} = x_3$

**Table 7:** (Continued.)

<b>g</b>	<b><math>\tilde{g}</math></b>	Non-zero Poisson brackets relations
	$(II \oplus R).xvii$	$\{x_1, x_2\} = -x_3, \{x_1, x_4\} = -x_3 - \frac{1}{2}x_3^2, \{x_2, x_4\} = x_3$
	$(II \oplus R).xviii$	$\{x_1, x_2\} = -x_3$
	$(II \oplus R).ii$	$\{x_1, x_2\} = x_4$
$(II \oplus R).iv$	$A_{4,7}$	$\{x_1, x_4\} = 2x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = -\frac{x_1^2}{2} + x_2, \{x_3, x_4\} = x_2 + x_3$
$(II \oplus R).ii$	$A_{4,7}$	$\{x_1, x_4\} = 2x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = x_2, \{x_3, x_4\} = \frac{x_1^2}{2} + x_2 + x_3$
$(II \oplus R).x$	$A_{4,9}^0$	$\{x_1, x_4\} = x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = -\frac{x_1^2}{2} + x_2$
$(II \oplus R).viii$	$A_{4,9}^0$	$\{x_1, x_4\} = x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = -\frac{x_1^2}{2} + x_2, \{x_3, x_4\} = \frac{x_1^2}{2}$
$(II \oplus R).iv$	$A_{4,9}^0$	$\{x_1, x_4\} = x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = -\frac{x_1^2}{2} + x_2$
$(II \oplus R).ii$	$A_{4,9}^0$	$\{x_1, x_4\} = x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = x_2, \{x_3, x_4\} = \frac{x_1^2}{2}$
$(II \oplus R).xi$	$A_{4,9}^0$	$\{x_1, x_2\} = -\frac{x_1^2}{2}, \{x_1, x_4\} = x_1 - \frac{x_1^2}{2}, \{x_2, x_3\} = x_1 + \frac{x_1^2}{2}, \{x_2, x_4\} = x_2, \{x_3, x_4\} = -\frac{x_1^2}{2}$
$(II \oplus R).iv$	$A_{4,9}^{-\frac{1}{2}}$	$\{x_1, x_4\} = \frac{x_1}{2}, \{x_2, x_3\} = x_1, \{x_2, x_4\} = x_2 - \frac{x_1^2}{2}, \{x_3, x_4\} = -\frac{x_3}{2}$
$(II \oplus R).ii$	$A_{4,9}^{-\frac{1}{2}}$	$\{x_1, x_4\} = \frac{x_1}{2}, \{x_2, x_3\} = x_1, \{x_2, x_4\} = x_2, \{x_3, x_4\} = \frac{x_1^2 - x_3}{2}$
$(II \oplus R).ii$	$A_{4,9}^1$	$\{x_1, x_4\} = 2x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = x_2, \{x_3, x_4\} = \frac{x_1^2}{2} + x_3$
$(II \oplus R).iv$	$A_{4,9}^b$	$\{x_1, x_4\} = (1+b)x_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = -\frac{x_1^2}{2} + x_2, \{x_3, x_4\} = bx_3$
$(II \oplus R).ii$	$A_{4,11}^b$	$\{x_1, x_4\} = 2bx_1, \{x_2, x_3\} = x_1, \{x_2, x_4\} = bx_2 - x_3, \{x_3, x_4\} = \frac{x_1^2}{2} + x_2 + bx_3$
$(II \oplus R).xii$	$A_{4,12}$	$\{x_1, x_3\} = x_1, \{x_1, x_4\} = -x_2, \{x_2, x_3\} = x_2, \{x_2, x_4\} = x_1, \{x_3, x_4\} = \frac{x_2^2 - x_1^2}{2}$
$(II \oplus R).xiii$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = x_4$
$(III \oplus R).i$	$A_2 \oplus A_2$	$\{x_1, x_2\} = 1 - e^{-x_2}, \{x_3, x_4\} = x_4$
$(III \oplus R).iii$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = 1 - e^{-x_4}$
$(III \oplus R).iv$	$A_2 \oplus A_2$	$\{x_1, x_2\} = 1 - e^{-x_2}, \{x_3, x_4\} = x_4$
$(III \oplus R).v$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = 1 - e^{-x_4}$
$(VI_0 \oplus R).vi$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = x_4$
$(VI_0 \oplus R).vii$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = -1 + e^{x_4}$
$(VI_0 \oplus R).viii$	$A_2 \oplus A_2$	$\{x_1, x_2\} = x_2, \{x_3, x_4\} = -1 + e^{x_4}$

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